Chapter 2
Random Dynamical Systems with Inputs

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Abstract This work introduces a notion of random dynamical systems with inputs, providing several basic definitions and results on equilibria and convergence. It also presents a "converging input to converging state" ("CICS") result, a concept that plays a key role in the analysis of stability of feedback interconnections, for monotone systems.

Keywords Pullback convergence • Random dynamical systems • Stochastic dynamics

2.1 Introduction

In the late 1980s, Ludwig Arnold conceived an elegant and deep approach to the foundations of random dynamics [3]. His paradigm of a *random dynamical system* (RDS for short) is based on an ultimately simple idea: view an RDS as consisting of two ingredients, a stochastic but autonomous "noise process," and a classical dynamical system that is driven by this process. The noise process is described by a measure-preserving dynamical system. It is typically probabilistic, representing for example environmental perturbations, internal variability, randomly fluctuating parameters, model uncertainty, or measurement errors. But the formalism allows for deterministic periodic or almost-periodic driving processes as well. The resulting theory, developed since by many authors, provides a seamless integration of classical ergodic theory with modern dynamical systems, giving a theoretical framework parallel to classical smooth and topological dynamics (stability, attractors, bifurcation theory, and so forth), while allowing one to treat ina unified way the most

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important classes of dynamical systems with randomness—random differential or difference equations (basically, deterministic systems with randomly changing parameters), or stochastic ordinary and partial differential equations (white noise or, more generally, martingale-driven systems as studied in the Itô calculus). The main goal of this chapter is to propose a new RDS-based formalism for random control systems, that is, systems with inputs (and outputs), which we abbreviate RDSI (or RDSIO).

Why Systems with Inputs and Outputs?

Our motivation for studying RDS with inputs and outputs arises from the need to provide foundations for a constructive theory of interconnections and feedback for stochastic systems, one that will eventually generalize successful and widely applied deterministic approaches to the analysis and design of dynamic networks [17, 19, 20]. To motivate this need and in order to set the stage for our definitions, let us start by recalling the basic paradigm of (deterministic) control theory. We use for concreteness ordinary differential equations. (For a more abstract general dynamical systems approach, see [30], as well as the definition of RDSIO's in this chapter.) The objects of study are systems with inputs and outputs:

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\
&\ \vdots \\
\dot{x}_n(t) &= f_n(\underbrace{x_1(t), \ldots, x_n(t)}_{\text{states}}, \underbrace{u_1(t), \ldots, u_m(t)}_{\text{inputs}})
\end{aligned}
$$

supplemented by a set of output variables $y_1, \ldots, y_p$ that are functions of the state vector x:

$$
y_j(t) = h_j(x(t)), \quad j = 1, \ldots, p.
$$

The inputs $u_i(t)$ may be viewed as controls, forcing functions, external signals, or stimuli, depending on the context. The outputs y_j represent responses, typically a partial read-out of the system state vector $(x_1, \ldots, x_n)$. Such a formalism, which originated in the analysis of engineering systems, is also natural in biology. Cells are not autonomous systems; they process external information, provided by physical (UV or other radiation, mechanical, temperature) or chemical (drugs, growth factors, hormones, nutrients) inputs. They also produce signals which we may view as outputs, such as chemical signals sent to other cells, commands to motors that move flagella or pseudopods, or the internal activation of transcription factors which may be monitored by measurement technologies. Thus, the control-theory formalism—in contrast to dynamical-systems theory, which deals with isolated systems—is not only reasonable, but natural in biology.

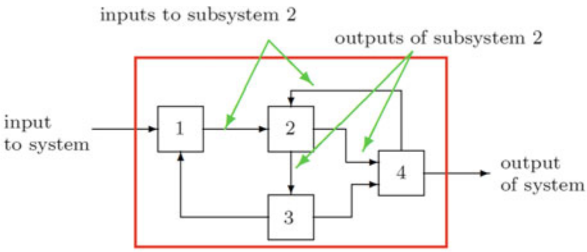

Fig. 2.1 A system viewed as an interconnection of subsystems with inputs and outputs

There is also a somewhat different reason for considering systems with inputs and outputs. Cells can be seen as composed of a large number of subsystems, networks of proteins, RNA, DNA, and metabolites involved in various processes such as cell growth and maintenance, division, and death. Indeed, one of the important themes in current molecular biology [9,15,22] is that of understanding cell behavior in terms of cascades and feedback interconnections of elementary "modules." The hope is that one should be able to decompose large systems into, hopefully simpler, subsystems, and then study the emergent properties of interconnections. Diagrammatically, one might represent this situation by a graph as in Fig. 2.1, which shows an overall system as composed of four subsystems. In Fig. 2.1, there are inputs and outputs for the overall system. However, even if the entire system were autonomous (no arrows into or out of the large box), in order to be able to define such interconnections, one must necessarily consider subsystems that admit time-dependent input signals and which produce output signals. Thus, the control theoretic formalism is a necessity even in the analysis of autonomous systems, when using a decomposition-based approach. Observe that, if the behavior of subsystems is subject to random effects, then it is imperative to allow inputs to be random when studying subsystems: for example, the subsystem "2" in Fig. 2.1 has inputs that depend on subsystems "1" and "4" and thus, if these are described by random processes, the inputs to "2" are also random processes.

As an illustration of how these ideas play out in the deterministic case, consider an inhibitory or activating cyclic structure

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_n, x_1) \\
\dot{x}_2 &= f_2(x_1, x_2) \\
&\vdots \\
\dot{x}_n &= f_n(x_{n-1}, x_n),
\end{aligned}
$$

as diagrammed in the left panel of Fig. 2.2. This is the "Goodwin model" of gene expression, and appears as well in many other models in mathematical biology (e.g. [14, 25]). It has been much studied mathematically, notably by Mallet-Paret and others [13,24,27,28], which among other major results, established a Poincaré-Bendixson theorem which tightly characterizes Ω-limit sets for such systems in

Fig. 2.2 A cyclic system (*left*), built by feedback from a cascade of n systems (*right*)

terms of periodic orbits and heteroclinic connections among equilibria. In the present context, we wish to view the system as built out of n components x_i, $i = 1, \ldots, n$. These components form a "cascade" or "series interconnection" when the feedback connection is ignored (right panel of Fig. 2.2). This view has been very successful when combined with tools from passivity theory [2], input-to-state (ISS) stability [29], and monotone systems with inputs and outputs [1]. To be more concrete, suppose, for example, that the system has the following special form, with each x_i scalar:

$$\begin{aligned} \dot{x}_1 &= b_1\kappa_1(x_n) - a_1x_1 \\ \dot{x}_2 &= b_2\kappa_2(x_1) - a_2x_2 \\ &\vdots \\ \dot{x}_n &= b_n\kappa_n(x_{n-1}) - a_nx_n, \end{aligned}$$

where the a_i's and b_i's are (for the moment) positive constants. The functions $\kappa_i(x_{i-1})$ represent the way in which the previous state in the cycle affects the given state. "Opening up" the feedback loop amounts to studying the system:

$$\begin{aligned} \dot{x}_1 &= b_1\kappa_1(u) - a_1x_1 \\ \dot{x}_2 &= b_2\kappa_2(x_1) - a_2x_2 \\ &\vdots \\ \dot{x}_n &= b_n\kappa_n(x_{n-1}) - a_nx_n, \end{aligned}$$

in which now u represents an external input. We may, in turn, view this open system as an interconnection of n subsystems

$$\dot{x} = b_i\kappa_i(u) - a_ix.$$

The hope is to be able to conclude something interesting about the overall system by the following two steps: (1) study the "open" system by recursively interconnecting the systems $\dot{x} = b_i\kappa_i(u) - a_ix$ until the whole system is obtained, and then (2) study the effect of "closing the loop" with feedback to recover the original system. The key property needed in the first step, at least in order to recursively study stability, is the CICS property: the state $x_i(t)$ should converge to an equilibrium provided that the input $u(t)$ converges to a limit. Obviously, in this simple example CICS is trivially true (assuming that κ is continuous), since we just have a forced linear system, easily solved in closed form using variation of parameters. However, for general nonlinear systems, CICS fails even for systems which are globally asymptotically stable with respect to constant inputs. This motivated, for deterministic systems, the introduction of the notions of ISS [29] and of monotone systems with inputs [1], either of which allows one to obtain CICS types of theorems, and these approaches coupled with what are generically called "small-gain theorems" (essentially, asking that the feedback loop results in a contraction in an appropriate sense) allow one to complete the program (step 2). In this work, we focus exclusively on the CICS problem (step 1) for stochastic systems, and leave the study of small-gain theorems for follow-up work.

Stochastic extensions of deterministic theory should take full advantage of the power of ergodic theory. Suppose, continuing with the above simple example, that we have the scalar linear system $\dot{x} = bu - ax$, where a and b are now not constants but are randomly varying, $a = a(\omega)$, $b = b(\omega)$. Randomness might model the effect of cell-to-cell variability in essential enzymes, or physical factors such as temperature or pH. If $a(\omega) \leq -\lambda < 0$ for all ω (and b is, for example, bounded), then stability will not be an issue. However, it may be that the only possible assumption is that the expected value of $a(\omega)$ is negative, but $a(\omega)$ might take zero, or even positive, values (for example, a might be a difference between an auto-catalytic term of production and a degradation/dilution term). Then, ergodic theory is needed in order to establish results on almost-sure stability (or convergence to steady-state probability distributions). We feel, therefore, that an RDS-based theory is most natural in this context.

Much work has been done on random control systems, but not employing an RDS axiomatic approach. This includes the papers [11, 26] on stochastic stabilization, as well as the papers [7, 8, 31] on feedback stabilization using noise to state stability analogs of input to state stability. We believe, however, that an RDS approach is a useful addition to the literature, for the reasons mentioned above. Also very relevant is an extension [6] of RDS to allow (*deterministic*) inputs that are themselves generated by a dynamical system (in the terminology of regulation and disturbance rejection, one would say that inputs are generated by an "exosystem").

Outline of Chapter

We first review the classical RDS theory. This material is not new; however, with an eye to generalizations, we reformulate it in a slightly different language. We next define our new concept of RDSI (and RDSIO), which extends the notion of RDS to systems in which there is an external input or forcing function, which is itself a stochastic process. A major contribution of this work lies upon the precise formulation of this concept, particularly the way in which the stochastic argument of the input is shifted in the semigroup (cocycle) property. Note that stochasticity of inputs is essential if one is to develop a theory of interconnected subsystems, as an input to one system in such an interconnection is typically obtained by using a combination of outputs (necessarily random) of other subsystems. After establishing several basic results that provide a foundation for further study, we turn to the question of "converging input to converging state" (CICS) properties. Specifically, recent work by Chueshov [5] introduced the class of monotone RDS (without inputs), a theory that provides us with the concepts needed to pursue the generalization of the latter to RDSI. Thus, we introduce also a class of monotone RDSI, and are able to formulate and prove a CICS theorem for monotone systems. A follow-up of this work will provide a small-gain theorem for monotone RDSI, generalizing [1], which follows from the CICS tools developed here. Separate work in progress deals with generalizations of ISS. Space prevents giving many examples, so we limit ourselves to a simple linear RODE (a pathwise random ODE). In principle, however, our setup also allows one to study more complicated objects including stochastic differential equations as in the Itô calculus. (A good reference for RODE's and SDE's in the context of RDS is the original book by Arnold [3]; see also [18]).

Other chapters in this volume deal with concepts closely related to those discussed in this chapter. Linear systems with inputs are considered, for example, in Chap. 1, Example 1.4, when viewing the transcription factor activity $f(t)$ as an input. Pullback limits are discussed in Example 1.7 of that same chapter, and especially at the end of Sect. 1.6, where the significance of this concept is discussed. Cascade flows (semi-direct products, skew-product flows) are described in Sect. 1.9. The mass-action kinetics model of the JAK-STAT signal transduction pathway described in Chap. 9, (9.7), can also be interpreted as a cascade closed under the feedback of x_4 into the first coordinate. It is in fact a monotone system. Finally, the base model given in Chap. 8, Sect. 8.2.6 for hepatitis C virus viral kinetics in chronically infected patients, can be interpreted as a closed-loop system. More specifically, it can be viewed as the closed-loop obtained from a monotone stochastic RDS (with cone $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}$, and when the term $T(t)$ in the equation for $I(t)$ is viewed as an input), closed under "negative" feedback, when setting this input again to $T(t)$.

2.2 Random Dynamical Systems

We first review the random dynamical systems framework of Arnold [3]. Along the way we introduce a couple of pieces of terminology not found in [3], to facilitate the discussion. Suppose given a *measure preserving dynamical system*[1] (*MPDS*)

$$\theta = (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathscr{T}});$$

[1] Arnold [3, p. 635] and Chueshov [5, p. 10, Definition 1.1.1] refer to such an object primarily as a *metric dynamical system*. We find *measure preserving*, which Arnold also uses as a synonym, less confusing and more informative.

that is, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a topological group $(\mathscr{T}, +)$, and a measurable flow $\{\theta_t\}_{t\in\mathscr{T}}$ of measure preserving maps $\Omega \to \Omega$ satisfying (T1)–(T3):

(T1) $(t,\omega) \mapsto \theta_t\omega$, $(t,\omega) \in \mathscr{T} \times \Omega$, is $(\mathscr{B}(\mathscr{T}) \otimes \mathscr{F})$-measurable,
(T2) $\theta_{t+s} = \theta_t \circ \theta_s$ for every $t, s \in \mathscr{T}$ (semigroup property),
(T3) $\mathbb{P} \circ \theta_t = \mathbb{P}$ for each $t \in \mathscr{T}$ (measure preserving[2]).

In this work $\mathscr{T}$ will always refer to either $\mathbb{R}$ or $\mathbb{Z}$, depending on whether one is talking about continuous or discrete time, respectively. In either case $\mathscr{T}_{\geq 0}$ refers to the nonnegative elements of $\mathscr{T}$. We will occasionally need to make measure-theoretic considerations about $\mathscr{T}$ or Borel subsets of it. If $\mathscr{T} = \mathbb{R}$, that is, in continuous time, then we tacitly equip any Borel subset of $\mathscr{T}$ with the measure induced by the Lebesgue measure on $\mathbb{R}$. If $\mathscr{T} = \mathbb{Z}$, or in discrete time, then we think of the counting measure in $\mathbb{Z}$. When $\mathscr{T} = \mathbb{Z}$, it follows from (T2) that θ is completely determined by $\theta_1 = \theta(1,\cdot)$. In that case we will abuse the notation and use the same θ to denote both the underlying MPDS and θ_1.

In the context of a given MPDS θ, a set $B \in \mathscr{F}$ is said to be θ-invariant if $\theta_t(B) = B$ for all $t \in \mathscr{T}$. We say that an MPDS θ is *ergodic* (*under* $\mathbb{P}$) if, whenever $B \in \mathscr{F}$ is θ-invariant, then we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

Let X be a metric space constituting the measurable space $(X, \mathscr{B})$ when equipped with the σ-algebra $\mathscr{B}$ of Borel subsets of X. A (*continuous*) *random dynamical system* (*RDS*) *on* X is a pair (θ, φ) in which θ is an MPDS and

$$\varphi : \mathscr{T}_{\geq 0} \times \Omega \times X \longrightarrow X$$

is a (*continuous*) *cocycle over* θ; that is, a $(\mathscr{B}(\mathscr{T}_{\geq 0}) \otimes \mathscr{F} \otimes \mathscr{B})$-measurable map such that

(S1) $\varphi(t,\omega) := \varphi(t,\omega,\cdot) : X \to X$ is continuous for each $t \in \mathscr{T}_{\geq 0}$, $\omega \in \Omega$,
(S2) $\varphi(0,\omega) = \mathrm{id}_X$ for each $\omega \in \Omega$, and (cocycle property)

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega), \quad \forall s,t \in \mathscr{T}_{\geq 0},\ \forall \omega \in \Omega.$$

The cocycle property generalizes the semigroup property of deterministic dynamical systems. More specifically, RDS's include deterministic dynamical systems as the special case in which Ω is a singleton.

Example 2.1 (RDS's Generated by Random Linear Differential Equations). Given an MPDS θ, suppose $A\colon \Omega \to \mathbb{R}^{n\times n}$ is a random $n \times n$ real matrix such that, for each $\omega \in \Omega$,

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[2] Property (T3) is normally [32, Definition 1.1] stated as

$$\mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}(B), \quad \forall B \in \mathscr{F},\ \forall t \in \mathscr{T}.$$

But since it follows from (T2) that θ_t is invertible with $\theta_t^{-1} = \theta_{-t}$ for each $t \in \mathscr{T}$, the two formulations are equivalent in this context.

$$t \longmapsto \|A(\theta_t \omega)\|, \quad t \geq 0,$$

is locally essentially bounded. For each $\omega \in \Omega$, let

$$\Xi(\cdot, \cdot, \omega) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$$

be the *fundamental matrix solution*[3] of the linear differential equation

$$\dot{\xi} = A(\theta_t \omega)\xi, \quad t \in \mathbb{R}; \tag{2.1}$$

that is, for each fixed $s \in \mathbb{R}$, $\Xi(s, \cdot, \omega)$ is the unique absolutely continuous $\mathbb{R} \to \mathbb{R}^{n \times n}$ map such that

$$\Xi(s, s, \omega) = I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$\frac{d}{dt}\Xi(s, t, \omega) = A(\theta_t \omega)\Xi(s, t, \omega)$$

for almost all $t \in \mathbb{R}$.

Let

$$\begin{aligned} \Phi \colon \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t, \omega, x) &\longmapsto \Xi(0, t, \omega) \cdot x \end{aligned}.$$

Then $\Phi(0, \omega, x) = x$ for every $(\omega, x) \in \Omega \times \mathbb{R}^n$ and

$$\frac{d}{dt}\Phi(t, \omega, x) = A(\theta_t \omega)\Phi(t, \omega, x)$$

for almost all $t \geq 0$. Moreover, $\Phi(t, \omega, \cdot) \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous for each fixed $(t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega$, and it can be shown using existence and uniqueness of solutions for (2.1) that Φ has the cocycle property:

$$\Phi(t + s, \omega, x) = \Phi(t, \theta_s \omega, \Phi(t, \omega, x)), \quad \forall (t, \omega, x) \in \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^n.$$

Thus (θ, Φ) constitutes an RDS, referred to as the RDS *generated* by the (*homogeneous, linear*) *random differential equation* (*RDE*) (2.1).

[3]The reason we are introducing the fundamental matrix solution as a function of $(s, t) \in \mathbb{R} \times \mathbb{R}$ rather than a function of just $t \in \mathbb{R}$ (for each fixed $\omega \in \Omega$) will become clear in Example 2.3. This notation will make it easier to discuss the rate of growth of the fundamental matrix solution.

In this work, we use linear (or affine) systems as a case study to illustrate the theory developed. Such systems (and their discrete counterparts) may be interpreted as "switched linear systems," and include classes of systems of great interest in applications such as iterated function systems. Throughout the remainder of the chapter, we will be building upon the example above. Thus for any random matrix A as in Example 2.1, the symbols "Ξ" and "Φ" will be reserved to carry the meanings established in the example. We shall need the following two properties of the fundamental matrix solution:

(F1) $\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1} = \Xi(s,t,\omega)$, for all $(s,t,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega$, and
(F2) $\Xi(s,t,\theta_\sigma\omega) = \Xi(\sigma+s,\sigma+t,\omega)$, for all $(s,t,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega$, for all $\sigma\in\mathbb{R}$.

These properties also follow from uniqueness of solutions.

2.2.1 Trajectories, Equilibria and θ-Stationary Processes

In the context of RDS's, the analogue to points in the state space X for a deterministic system are random variables $\Omega\to X$, that is, $\mathscr{B}$-measurable maps $\Omega\to X$. We denote the set of all random variables on a metric space X by $X^{\Omega}_{\mathscr{B}}$. We refer to a $(\mathscr{B}(\mathscr{T}_{\geq 0})\otimes\mathscr{F})$-measurable map $q:\mathscr{T}_{\geq 0}\times\Omega\to X$ as a *θ-stochastic process*[4] *on X*, and denote $q_t := q(t,\cdot)$ for each $t\in\mathscr{T}_{\geq 0}$. The set of all θ-stochastic processes on a metric space X is denoted by $\mathscr{S}^X_\theta$.

Let (θ,φ) be an RDS. Given $x\in X^{\Omega}_{\mathscr{B}}$, we define the (*forward*) *trajectory starting at x* to be the θ-stochastic process $\xi^x\in\mathscr{S}^X_\theta$ defined by

$$\xi^x_t(\omega) := \varphi(t,\omega,x(\omega)),\quad (t,\omega)\in\mathscr{T}_{\geq 0}\times\Omega. \tag{2.2}$$

The *pullback trajectory starting at x* is in turn defined to be the θ-stochastic process $\check{\xi}^x:\mathscr{T}_{\geq 0}\times\Omega\to X$ defined by

$$\check{\xi}^x_t(\omega) := \varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega)),\quad (t,\omega)\in\mathscr{T}_{\geq 0}\times\Omega. \tag{2.3}$$

More generally, the *pullback* of a θ-stochastic process $q\in\mathscr{S}^X_\theta$ is the θ-stochastic process $\check{q}\in\mathscr{S}^X_\theta$ defined by

$$\check{q}_t(\omega) := q_t(\theta_{-t}\omega),\quad (t,\omega)\in\mathscr{T}_{\geq 0}\times\Omega.$$

So the pullback trajectory starting at x is simply the pullback of the forward trajectory starting at x. We will always use the accent ˇ to indicate the pullback of the θ-stochastic process being accented.

[4] A "θ-stochastic process" is indeed a stochastic process in the traditional sense. We use the prefix "θ-" to emphasize the underlying probability space, as well as the time semigroup.

We slightly modify the standard notion of equilibrium for RDS's (see, for instance, [5, p. 38, Definition 1.7.1]) to allow for the defining property to hold only almost everywhere, as opposed to everywhere. So an *equilibrium* of an RDS (θ, φ) is a random variable $x \in X^{\Omega}_{\mathscr{B}}$ such that

$$\xi^x_t(\omega) = \varphi(t, \omega, x(\omega)) = x(\theta_t \omega), \quad \forall t \in \mathscr{T}_{\geq 0}, \ \forall \omega \in \tilde{\Omega},$$

for some θ-invariant $\tilde{\Omega} \subseteq \Omega$ of full measure.[5] It is often not necessary to specify the said $\tilde{\Omega}$. So we say "for θ-almost all $\omega \in \Omega$" and write

$$\text{`}\ \tilde{\forall}\omega \in \Omega\ \text{'}$$

to mean "for all $\omega \in \tilde{\Omega}$, for some θ-invariant set $\tilde{\Omega} \subseteq \Omega$ of full measure."

In view of the notion of pullback convergence with which we will be working (see Sect. 2.2.3), it is more natural to think of the concept of equilibrium in terms of pullback trajectories. Observe that a random variable $x \in X^{\Omega}_{\mathscr{B}}$ is an equilibrium of the RDS (θ, φ) if, and only if

$$\check{\xi}^x_t(\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = x(\omega), \quad \forall t \in \mathscr{T}_{\geq 0}, \ \tilde{\forall}\omega \in \Omega.$$

The remaining of this section is devoted to interpreting the concept of equilibrium for an RDS in terms of a shift operator in the set $\mathscr{S}^X_\theta$ of all θ-stochastic processes on X. For each $s \in \mathscr{T}_{\geq 0}$, let

$$\begin{aligned} \rho_s : \mathscr{S}^X_\theta &\longrightarrow \mathscr{S}^X_\theta \\ q &\longmapsto \rho_s(q) \end{aligned} \tag{2.4}$$

be defined by

$$(\rho_s(q))_t(\omega) := q_{t+s}(\theta_{-s}\omega), \quad (t, \omega) \in \mathscr{T}_{\geq 0} \times \Omega. \tag{2.5}$$

Definition 2.1 (θ-Stationary Process). A θ-stochastic process $\bar{q} \in \mathscr{S}^X_\theta$ is said to be θ-*stationary* if

$$(\rho_s(\bar{q}))_t(\omega) = \bar{q}_t(\omega),$$

for all $s, t \in \mathscr{T}_{\geq 0}$, for θ-almost all $\omega \in \Omega$.

We use the prefix "θ-" in "θ-stationary" to emphasize the dependence on the underlying MPDS θ. Using the characterization of θ-stationary processes given in

[5] That is, $\theta_t \tilde{\Omega} = \tilde{\Omega}$ for all $t \in \mathscr{T}$, and $\mathbb{P}(\tilde{\Omega}) = 1$.

Lemma 2.1 below, it is not difficult to show that a θ-stationary θ-stochastic process $\bar{q}$ is indeed *stationary* in the traditional stochastic processes sense:

$$\mathbb{P}(\bar{q}_{t_1} \in A_1, \ldots, \bar{q}_{t_k} \in A_k) = \mathbb{P}(\bar{q}_{t_1+h} \in A_1, \ldots, \bar{q}_{t_k+h} \in A_k)$$

for all $A_1, \ldots, A_k \in \mathscr{F}$, for any $t_1, \ldots t_k, h \geq 0$ (see, for instance, [23, Sect. 1.3]).

Lemma 2.1. *The θ-stochastic process $\bar{q} \in \mathscr{S}_\theta^X$ is θ-stationary if and only if there exists a random variable $q \in X_{\mathscr{B}}^{\Omega}$ such that*

$$\bar{q}_t(\omega) = q(\theta_t \omega), \quad \forall t \in \mathscr{T}_{\geq 0},\ \tilde{\forall} \omega \in \Omega. \tag{2.6}$$

Proof. (Sufficiency) Suppose that (2.6) holds for some $q \in X_{\mathscr{B}}^{\Omega}$. Pick any $s \in \mathscr{T}_{\geq 0}$. For any $t \in \mathscr{T}_{\geq 0}$ and θ-almost all $\omega \in \Omega$,

$$(\rho_s(\bar{q}))_t(\omega) = \bar{q}_{t+s}(\theta_{-s}\omega) = q(\theta_{t+s}\theta_{-s}\omega) = q(\theta_t\omega) = \bar{q}_t(\omega).$$

So $\bar{q}$ is θ-stationary.

(Necessity) Suppose that $\bar{q} \in \mathscr{S}_\theta^X$ is θ-stationary and define $q \in X_{\mathscr{B}}^{\Omega}$ by

$$q(\omega) := \bar{q}_0(\omega), \quad \omega \in \Omega. \tag{2.7}$$

We have

$$\bar{q}_{t+s}(\theta_{-s}\omega) = (\rho_s(q))_t(\omega) = \bar{q}_t(\omega), \quad \forall s, t \in \mathscr{T}_{\geq 0},\ \tilde{\forall} \omega \in \Omega.$$

Setting $t = 0$ and renaming s as t we then have

$$\bar{q}_t(\theta_{-t}\hat{\omega}) = \bar{q}_0(\hat{\omega}) = q(\hat{\omega}), \quad \forall t \in \mathscr{T}_{\geq 0},\ \tilde{\forall} \hat{\omega} \in \Omega.$$

Given any $\omega \in \tilde{\Omega}$ and any $t \in \mathscr{T}_{\geq 0}$, we may apply this property with $\hat{\omega} = \theta_t\omega$ due to the θ-invariance of $\tilde{\Omega}$, thus obtaining

$$\bar{q}_t(\omega) = q(\theta_t\omega).$$

Therefore (2.6) holds. □

Note that the random variable q associated to $\bar{q}$ is unique up to a θ-invariant set of measure zero. Indeed, it is determined θ-almost everywhere by (2.7). Thus, we have:

Corollary 2.1. *Given an RDS (θ, φ) over a metric space X and a random state $x \in X_{\mathscr{B}}^{\Omega}$, the following three properties are equivalent:*

(1) x is an equilibrium;
(2) the trajectory ξ^x, as defined in (2.2), is θ-stationary;
(3) the map $t \mapsto \check{\xi}_t^x \in X_{\mathscr{B}}^{\Omega}$, $t \in \mathscr{T}_{\geq 0}$, is constant.

We will always use an overbar to denote the θ-stationary θ-stochastic process $\bar{q}$ associated with a given random variable q.

2.2.2 Perfection of Crude Cocycles

We briefly review the theory of perfection of crude cocycles discussed in Arnold's [3, Sect. 1.2]. It is customary for the definition of an RDS to require that the cocycle property of φ in (S2) holds for every $s, t \in \mathscr{T}_{\geq 0}$ and *every* $\omega \in \Omega$. If we want to emphasize this fact we shall say that φ is a *perfect cocycle* (over the underlying MPDS θ).

Definition 2.2 (Crude Cocycle). We say that $\varphi\colon \mathscr{T}_{\geq 0} \times \Omega \times X \to X$ is a *crude cocycle* (*over* θ) if it is a $(\mathscr{B}(\mathscr{T}) \otimes \mathscr{F} \otimes \mathscr{B})$-measurable map satisfying (S1) and

(S2′) $\varphi(0, w) = \mathrm{id}_X$ for each $\omega \in \Omega$, and for every $s \in \mathscr{T}_{\geq 0}$, there exists a subset $\Omega_s \subseteq \Omega$ of full measure such that

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \forall t \in \mathscr{T}_{\geq 0}, \ \forall \omega \in \Omega_s.$$

The Ω_s's need not be θ-invariant.

As Arnold points out, there are circumstances where this flexibility in the requirements for a cocycle is desirable. For instance, the flow of a stochastic differential equation is only guaranteed to be a crude cocycle [3, Sect. 2.3]. Another example will come up below after we introduce random dynamical systems with inputs. Consider (deterministic) controlled dynamical systems. Such systems yield a (deterministic) dynamical system when restricted to a constant input. One would expect a sensible extension of the concept to random dynamical systems to have an analogous property. However we shall see in the proof of Lemma 2.3 in the next section that the restriction of the flow of an RDS with inputs to a θ-stationary input is not necessarily a perfect cocycle.

In this work we deal only with random dynamical systems (with inputs) evolving in locally compact, connected subsets of $\mathbb{R}^n$. We will informally refer to such systems as *finite dimensional*. It turns out that crude cocycles evolving in these spaces can be perfected in a very reasonable sense.

Definition 2.3 (Indistinguishable Cocycles). Let θ be an MPDS and $\varphi, \psi\colon \mathscr{T}_{\geq 0} \times \Omega \times X \to X$ crude cocycles over θ. If there exists a subset $N \in \mathscr{F}$ such that $\mathbb{P}(N) = 0$ and

$$\{\omega \in \Omega;\ \varphi(t, \omega) \neq \psi(t, \omega),\ \text{for some } t \in \mathscr{T}_{\geq 0}\} \subseteq N,$$

then φ and ψ are said to be *indistinguishable*.

Proposition 2.1. *Let $\theta = (\mathscr{F}, \Omega, \mathbb{P}, (\theta_t)_{t\in\mathscr{T}})$ be an MPDS with $\mathscr{T} = \mathbb{Z}$ or $\mathscr{T} = \mathbb{R}$. Suppose $\varphi\colon \mathscr{T}_{\geq 0} \times \Omega \times X \to X$ is a crude cocycle over θ evolving in a locally compact, locally connected, Hausdorff topological space X. Then there exists a perfect cocycle $\psi\colon \mathscr{T}_{\geq 0} \times \Omega \times X \to X$ such that φ and ψ are indistinguishable.*

Proof. See Arnold [3, Theorem 1.2.1] for the discrete case, which actually holds with weaker hypotheses and yields stronger conclusions. For the continuous case, see Arnold [3, Theorem 1.2.2 and Corollary 1.2.4]. □

2.2.3 Pullback Convergence

We work with the notion of pullback convergence developed in the literature and canonized in the works of Arnold and Chueshov [3, 5]. As with equilibria, we relax the notion to require only that pointwise convergence happens θ-almost everywhere.

Definition 2.4 (Pullback Convergence). A θ-stochastic process $\xi \in \mathscr{S}_\theta^X$ is said to *converge to a random variable $\xi_\infty \in X_{\mathscr{B}}^{\Omega}$ in the pullback sense* if

$$\check{\xi}_t(\omega) := \xi_t(\theta_{-t}\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty,$$

for θ-almost all $\omega \in \Omega$.

Proposition 2.2. *Let (θ, φ) be an RDS evolving on a metric space X. Suppose there exists a random initial state $x \in X_{\mathscr{B}}^{\Omega}$ and a map $x_\infty : \Omega \to X$ such that*

$$\check{\xi}_t^x(\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) \longrightarrow x_\infty(\omega) \quad \textit{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega. \tag{2.8}$$

Then x_∞ is an equilibrium.

Proof. For each $t \in \mathscr{T}_{\geq 0}$, the map $\omega \mapsto \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))$, $\omega \in \Omega$, is measurable, since it is the composition of measurable maps:

$$\omega \longmapsto \theta_{-t}\omega \longmapsto x(\theta_{-t}\omega),$$

$$(\theta_{-t}\omega, x(\theta_{-t}\omega)) \longmapsto \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)).$$

So it follows from [21, Chap. 11, Sect. 1, Property M7 on page 248] that x_∞ is measurable. (If $\mathscr{T}$ is continuous time, just pick a subsequence $(t_n)_{n\in\mathbb{N}}$ going to infinity.)

In addition, for each $\omega \in \Omega$ such that the limit in (2.8) exists, and each $\tau \in \mathscr{T}_{\geq 0}$, we have

$$\lim_{t\to\infty} \varphi(t-\tau, \theta_{\tau-t}\omega, x(\theta_{\tau-t}\omega)) = x_\infty(\omega)$$

also. By θ-invariance, the limit in (2.8) exists for $\theta_\tau\omega$ as well. Hence

$$\begin{aligned} x_\infty(\theta_\tau\omega) &= \lim_{t\to\infty} \varphi(t, \theta_{-t}\theta_\tau\omega, x(\theta_{-t}\theta_\tau\omega)) \\ &= \lim_{t\to\infty} \varphi(\tau, \theta_{t-\tau}\theta\tau - t\omega, \varphi(t-\tau, \theta_{\tau-t}\omega, x(\theta_{\tau-t}\omega))) \\ &= \varphi(\tau, \omega, x_\infty(\omega)) \end{aligned}$$

by continuity (property (S1) in the definition of an RDS). □

2.3 RDS's with Inputs and Outputs

We now define a new concept. It extends the notion of RDS's to systems in which there is an external input or forcing function. A contribution of this work is the precise formulation of this concept, particularly the way in which the argument of the input is shifted in the semigroup (cocycle) property.

As in the previous section, given a metric space U, we equip it with its Borel σ-algebra $\mathscr{B}(U)$ and denote by $U^{\Omega}_{\mathscr{B}}$ the set of Borel measurable maps $\Omega \to U$. Let $\mathscr{S}^U_\theta$ be the set of all θ-stochastic processes $\mathscr{T}_{\geq 0} \times \Omega \to U$. Given $u, v \in \mathscr{S}^U_\theta$ and $s \in \mathscr{T}_{\geq 0}$, we define $u \diamondsuit_s v \colon \mathscr{T}_{\geq 0} \times \Omega \to U$ by

$$(u \diamondsuit_s v)_\tau(\omega) = \begin{cases} u_\tau(\omega),\ 0 \leq \tau < s \\ v_{\tau-s}(\theta_s\omega),\ s \leq \tau \end{cases}, \quad \tau \in \mathscr{T}_{\geq 0},\ \omega \in \Omega.$$

We say that a subset $\mathscr{U} \subseteq \mathscr{S}^U_\theta$ is a *set of* θ*-inputs* if $u \diamondsuit_s v \in \mathscr{U}$ for any $u, v \in \mathscr{U}$ and any $s \in \mathscr{T}_{\geq 0}$. In other words, a set of θ-inputs is a subset of $\mathscr{S}^U_\theta$ which is closed under concatenation.

Given $\tilde{u} \in U$, we denote by $c(\tilde{u})$ the trivial θ-stochastic process defined by $(c(\tilde{u}))_t(\omega) := \tilde{u}$ for every $t \in \mathscr{T}_{\geq 0}$ and every $\omega \in \Omega$.

Definition 2.5 (Random Dynamical Systems with Inputs). A *random dynamical system with inputs* (*RDSI*) is a triple $(\theta, \varphi, \mathscr{U})$ consisting of an MPDS

$$\theta = (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathscr{T}}),$$

a set of θ-inputs $\mathscr{U} \subseteq \mathscr{S}^U_\theta$, and a map

$$\varphi : \mathscr{T}_{\geq 0} \times \Omega \times X \times \mathscr{U} \to X$$

satisfying

(I1) $\varphi(\cdot,\cdot,\cdot,u)\colon \mathscr{T}_{\geq 0} \times \Omega \times X \to X$ is $(\mathscr{B}(\mathscr{T}_{\geq 0}) \otimes \mathscr{F} \otimes \mathscr{B})$-measurable for each fixed $u \in \mathscr{U}$;

(I1′) the map $\tilde{\varphi}: \mathscr{T}_{\geq 0} \times \Omega \times X \times U \to X$ defined by

$$\tilde{\varphi}(t, \omega, x, \tilde{u}) := \varphi(t, \omega, x, c(\tilde{u})), \quad (t, \omega, x, \tilde{u}) \in \mathscr{T}_{\geq 0} \times \Omega \times X \times U,$$

is $(\mathscr{B}(\mathscr{T}_{\geq 0}) \otimes \mathscr{F} \otimes \mathscr{B} \otimes \mathscr{B}(U))$-measurable;

(I2) $\varphi(t, \omega, \cdot, u) : X \to X$ is continuous, for each fixed $(t, \omega, u) \in \mathscr{T}_{\geq 0} \times \Omega \times \mathscr{U}$;

(I3) $\varphi(0, \omega, x, u) = x$ for each $(\omega, x, u) \in \Omega \times X \times \mathscr{U}$;

(I4) given $s, t \in \mathscr{T}_{\geq 0}$, $\omega \in \Omega$, $x \in X$, $u, v \in \mathscr{U}$, if

$$\varphi(s, \omega, x, u) = y$$

and

$$\varphi(t, \theta_s \omega, y, v) = z,$$

then

$$\varphi(s + t, \omega, x, u \diamondsuit_s v) = z;$$

(I5) and given $t \in \mathscr{T}_{\geq 0}$, $\omega \in \Omega$, $x \in X$, and $u, v \in \mathscr{U}$, if $u_\tau(\omega) = v_\tau(\omega)$ for almost all $\tau \in [0, t)$, then $\varphi(t, \omega, x, u) = \varphi(t, \omega, x, v)$.

We refer to the elements $u \in \mathscr{U}$ as *θ-inputs*, or simply *inputs*. Whenever we talk about an RDSI $(\theta, \varphi, \mathscr{U})$, we tacitly assume the notation laid above, unless otherwise specified.

(I1), (I1′) and (I2) are regularity conditions. (I3) means that nothing has "happened" if one is still at time $t = 0$. (I4) generalizes the cocycle property and (I5) states that the evolution of an RDS subject to an input u is, so to speak, independent of "irrelevant" random input values.

Remark 2.1. Notice that for each $s, t \in \mathscr{T}_{\geq 0}$, $x \in X$, $\omega \in \Omega$,

$$\varphi(t + s, \omega, x, u) \; = \; \varphi(t, \theta_s \omega, \varphi(s, \omega, x, u), \rho_s(u)), \quad \forall u \in \mathscr{U},$$

where $\rho_s: \mathscr{S}_\theta^U \to \mathscr{S}_\theta^U$ is defined by (2.5)[6]:

$$(\rho_s(u))_t(\theta_s \omega) \equiv u_{t+s}(\omega). \tag{2.9}$$

This follows from (I4) with $v = \rho_s(u)$, which then yields $u \diamondsuit_s v = u$. □

[6]We will use the same notation ρ_s for the shift operator $\mathscr{S}_\theta^V \to \mathscr{S}_\theta^V$ defined by (2.5), irrespective of the underlying metric space V. Since the domain of any θ-stochastic process is always $\mathscr{T}_{\geq 0} \times \Omega$, this will not be a source of confusion.

The shift operator ρ_s has a physical interpretation. The right-hand side is the input as interpreted by an observer of the RDSI φ who started at time $t_1 = 0$, while the left-hand side is how someone who started observing the system at time $t_2 = s$ would describe it at time t $(+\,t_2)$. Following this interpretation, a θ-stationary input would then be an input which is observed to be just the same, regardless of when one started observing it.

Example 2.2 (RDSI's Generated by Random Differential Linear Equations with Inputs). This generalizes Example 2.1. Given an MPDS θ, suppose that $A\colon \Omega \to \mathbb{R}^{n\times n}$ and $B\colon \Omega \to \mathbb{R}^{n\times k}$ are random real matrices such that, for each $\omega \in \Omega$,

$$t \longmapsto \|A(\theta_t\omega)\|, \quad t \geq 0, \quad \text{and} \quad t \longmapsto \|B(\theta_t\omega)\|, \quad t \geq 0,$$

are locally essentially bounded. Let $U := \mathbb{R}^k$ and let $\mathscr{S}_\infty^U \subseteq \mathscr{S}_\theta^U$ be the set of θ-inputs consisting of all θ-stochastic processes $u \in \mathscr{S}_\theta^U$ such that

$$t \longmapsto |u_t(\omega)|, \quad t \geq 0,$$

is locally essentially bounded for each $\omega \in \Omega$. We consider the *random differential equation with inputs* (*RDEI*)

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega), \quad t \geq 0,\ \omega \in \Omega,\ u \in \mathscr{S}_\infty^U. \tag{2.10}$$

Let $\Xi\colon \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ be the fundamental matrix solution of the homogeneous, linear RDE

$$\dot{\xi} = A(\theta_t\omega)\xi, \quad t \geq 0,$$

and let (θ, Φ) be the RDS generated by the same equation (see Example 2.1). For each fixed $(\omega, u) \in \Omega \times \mathscr{S}_\infty^U$, define

$$\Psi(\cdot, \omega, u)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^n$$

by

$$\Psi(t, \omega, u) := \int_0^t \Xi(\sigma, t, \omega) B(\theta_\sigma\omega) u_\sigma(\omega)\, d\sigma, \quad t \geq 0.$$

Finally, define

$$\begin{aligned} \varphi\colon \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^n \times \mathscr{S}_\infty^U &\longrightarrow \mathbb{R}^n \\ (t, \omega, x, u) &\longmapsto \Phi(t, \omega, x) + \Psi(t, \omega, u) \end{aligned}.$$

Fixing $(\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathscr{S}_\infty^U$ arbitrarily, and differentiating $\varphi(t, \omega, x, u)$ with respect to t, we get

$$\begin{aligned}\frac{d}{dt}\varphi(t,\omega,x,u) &= A(\theta_t\omega)\Phi(t,\omega,x) + \Xi(t,t,\omega)B(\theta_t\omega)u_t(\omega)\\ &\quad + A(\theta_t\omega)\int_0^t \Xi(\sigma,t\omega)B(\theta_\sigma\omega)u_\sigma(\omega)\,d\sigma\\ &= A(\theta_t\omega)\Psi(t,\omega,u) + B(\theta_t\omega)u_t(\omega), \quad \forall t \geq 0.\end{aligned}$$

Thus $t \to \varphi(t,\omega,x,u)$, $t \geq 0$, is a solution of (2.10) with initial state

$$\varphi(0,\omega,x,u) = \Phi(0,\omega,x) + \Psi(0,\omega,u) = x.$$

In fact, $(\theta,\varphi,\mathscr{S}^U_\infty)$ is an RDSI. Indeed, (I1) and (I1$'$) follow from the fact that the limit of a sequence of measurable functions is measurable. Properties (I2) and (I3) follow directly from the analogous properties of Φ. And (I4) and (I5) follow from uniqueness of solutions applied for each fixed $\omega \in \Omega$—one basically verifies that both sides of each equation we want to prove to be true, when looked at as functions of t, define solutions of the same differential equation with the same initial condition. We refer to $(\theta,\varphi,\mathscr{S}^U_\infty)$ as the RDSI *generated* by the RDEI (2.10).

We also introduce a notion of outputs.

Definition 2.6 (Random Dynamical System with Inputs and Outputs). A *random dynamical system with inputs and outputs* (*RDSIO*) is a quadruple $(\theta,\varphi,\mathscr{U},h)$, such that $(\theta,\varphi,\mathscr{U})$ is an RDSI, and

$$h : \Omega \times X \to Y$$

is an $(\mathscr{F}\otimes\mathscr{B})$-measurable map into a metric space Y such that $h(\omega,\cdot)$ is continuous for each $\omega \in \Omega$. In this context we call h an *output function* and Y an *output space*.

It may sometimes be useful to refer to a *random dynamical system with outputs* (*RDSO*) only, by which we mean a triple (θ,φ,h) where (θ,φ) is an RDS and h is an output function.

The Ω-component in the domain of output functions is important. It allows for the concept to model uncertainties in the readout as well. We will return to systems with outputs further down, in the context of RDSIO's which can be realized as cascades of RDSO's and RDSIO's.

2.3.1 Pullback Trajectories

Let $(\theta,\varphi,\mathscr{U},h)$ be an RDSIO with output space Y. Given $x \in X^\Omega_\mathscr{B}$ and $u \in \mathscr{U}$, we define the (*forward*) *trajectory starting at x and subject to u* to be the θ-stochastic process $\xi^{x,u} \in \mathscr{S}^X_\theta$ defined by

$$\xi^{x,u}_t(\omega) := \varphi(t,\omega,x(\omega),u), \quad (t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega.$$

We then define the *pullback trajectory starting at x and subject to u* to be the θ-stochastic process $\check{\xi}^{x,u} \in \mathscr{S}_\theta^X$ defined by

$$\check{\xi}_t^{x,u}(\omega) := \xi_t^{x,u}(\theta_{-t}\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u), \quad (t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega.$$

The (*forward*) *output trajectory corresponding to initial state x and input u* is defined to be the θ-stochastic process $\eta^{x,u} \in \mathscr{S}_\theta^Y$, where

$$\eta_t^{x,u}(\omega) := h(\theta_t\omega, \varphi(t,\omega, x(\omega), u)) = h(\theta_t\omega, \xi_t^{x,u}(\omega)), \quad (t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega,$$

while the *pullback output trajectory corresponding to initial state x and input u* is analogously defined to be the θ-stochastic process $\check{\eta}^{x,u} \in \mathscr{S}_\theta^Y$, where

$$\begin{aligned} \check{\eta}_t^{x,u}(\omega) &:= \eta_t^{x,u}(\theta_{-t}\omega) \\ &= h(\omega, \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)) \\ &= h(\omega, \check{\xi}_t^{x,u}(\omega)), \qquad (t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega. \end{aligned}$$

For RDSI's the definitions of forward and pullback trajectories are the same and we also use the notations $\xi^{x,u}$ and $\check{\xi}^{x,u}$. For RDSO's the definitions are analogous, except that they of course do not depend on any inputs. So forward and pullback trajectories are defined as for RDS's and we also use the notations ξ^x and $\check{\xi}^x$, respectively. We denote the forward and pullback output trajectories corresponding to initial state x by η^x and $\check{\eta}^x$, respectively:

$$\eta_t^x(\omega) := h(\theta_t\omega, \varphi(t,\omega, x(\omega))) = h(\theta_t\omega, \xi_t^x(\omega))$$

and

$$\check{\eta}_t^x(\omega) := h(\omega, \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))) = h(\omega, \check{\xi}_t^x(\omega))$$

for every $(t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega$.

Note that the input u is not shifted in the argument of φ in the pullback, while at first one might intuitively think it should have been. There are several reasons this is so. First notice that

$$\check{\xi}_t^{x,u}(\omega) = \xi_t^{x,u}(\theta_{-t}\omega), \quad \forall (t,\omega) \in \mathscr{T}_{\geq 0} \times \Omega.$$

So $\check{\xi}^{x,u}$ is just the pullback of the θ-stochastic process $\xi^{x,u}$, as it should be the case. However we are more concerned with what happens in the context of cascades and feedback interconnections of RDSIO's. But before we get to that we first discuss discrete RDSIO's. This will further motivate axioms (I1)–(I5) in the definition of an RDSI, provide—and completely characterize—a whole class of examples, and provide the framework for said discussion of pullback trajectories and cascades.

We say that an RDSI (or RDSIO) is *discrete* when $\mathscr{T} = \mathbb{Z}$. We first note that, just like RDS's [3, Sect. 2.1], RDSI's also have their flows completely determined by their state at time $t = 1$.

Theorem 2.1 (Characterization of Discrete RDSI's). *For every discrete RDSI*

$$(\theta, \varphi, \mathscr{U}),$$

there exists a unique map $f : \Omega \times X \times U \to X$ such that

(G1) $f : \Omega \times X \times U \to X$ *is* $(\mathscr{F} \otimes \mathscr{B} \otimes \mathscr{B}(U))$*-measurable,*
(G2) $f(\omega, \cdot, \tilde{u}) : X \to X$ *is continuous for each* $(\omega, \tilde{u}) \in \Omega \times U$,

and

$$\varphi(n+1, \omega, x, u) = f(\theta_n \omega, \varphi(n, \omega, x, u), u_n(\omega)), \tag{2.11}$$

for every $(n, \omega, x, u) \in \mathscr{T}_{\geq 0} \times \Omega \times X \times \mathscr{U}$.

Conversely, given an MPDS θ, a set of θ-inputs $\mathscr{U}$ and a map

$$f : \Omega \times X \times U \to X$$

satisfying (G1) *and* (G2)*, define* $\varphi : \mathscr{T}_{\geq 0} \times \Omega \times X \times \mathscr{U} \to X$ *recursively by*

$$\varphi(0, \omega, x, u) := x, \quad (\omega, x, u) \in \Omega \times X \times \mathscr{U}, \tag{2.12}$$

and (2.11)*. Then* $(\theta, \varphi, \mathscr{U})$ *is an RDSI.*

We refer to the map f as the generator *of the RDSI* $(\theta, \varphi, \mathscr{U})$.

Proof. Define f by setting

$$f(\omega, x, \tilde{u}) := \varphi(1, \omega, x, c(\tilde{u})), \quad (\omega, x, \tilde{u}) \in \Omega \times X \times U.$$

Then (G1) and (G2) follow directly from (I1′) and (I2), respectively. Equation (2.11) follows from (I4) (see Remark 2.1) and (I5):

$$\begin{aligned}
\varphi(n+1, \omega, x, u) &= \varphi(1, \theta_n \omega, \varphi(n, \omega, x, u), \rho_n(u)) \\
&= \varphi(1, \theta_n \omega, \varphi(n, \omega, x, u), c((\rho_n(u))_0(\theta_n \omega))) \\
&= f(\theta_n \omega, \varphi(n, \omega, x, u), (\rho_n(u))_0(\theta_n \omega)) \\
&= f(\theta_n \omega, \varphi(n, \omega, x, u), u_n(\omega))
\end{aligned}$$

for any $(n, \omega, x, u) \in \mathscr{T}_{\geq 0} \times \Omega \times X \times \mathscr{U}$. Uniqueness follows from (I3) and (I5), together with the computations above performed backwards for $t = 0$.

Now suppose f satisfies (G1) and (G2), and that φ is defined recursively by (2.12) and (2.11). For (I1), pick any $u \in \mathscr{U}$. One first shows using induction on n

that

$$\varphi(n,\cdot,\cdot,u) = f(\theta_{n-1}\cdot,\varphi(n-1,\cdot,\cdot,u),u_{n-1}(\cdot)) \tag{2.13}$$

is $(\mathscr{F}\otimes\mathscr{B})$-measurable for each $n\in\mathbb{Z}_{>0}$. Indeed, at $n=1$ we have

$$\varphi(1,\cdot,\cdot,u) = f(\theta_{1-1}\cdot,\varphi(1-1,\cdot,\cdot,u),u_{1-1}(\cdot)) = f(\cdot,\cdot,u_0(\cdot)),$$

which is $(\mathscr{F}\otimes\mathscr{B})$-measurable, since f satisfies (G1) and u_0 is $\mathscr{F}$-measurable. Now (2.13) gives us the inductive step, since the right hand side is a composition of measurable functions and, hence, itself measurable. Now pick any $A\in\mathscr{B}$. We then have

$$\varphi(\cdot,\cdot,\cdot,u)^{-1}(A) = \bigcup_{n=0}^{\infty}\{n\}\times\varphi(n,\cdot,\cdot,u)^{-1}(A)\in 2^{\mathbb{Z}_{\geq 0}}\otimes\mathscr{F}\otimes\mathscr{B},$$

since it is a countable union of $(2^{\mathbb{Z}_{\geq 0}}\otimes\mathscr{F}\otimes\mathscr{B})$-measurable sets. Thus (I1) holds. One can prove (I1′) in the same way by noting that

$$\tilde{\varphi}^{-1}(A) = \bigcup_{n=0}^{\infty}\{n\}\times\tilde{\varphi}(n,\cdot,\cdot,\cdot)^{-1}(A)$$

for each $A\in\mathscr{B}$, and that

$$\tilde{\varphi}(n,\cdot,\cdot,\cdot) = f(\theta_{n-1}\cdot,\tilde{\varphi}(n-1,\cdot,\cdot,\cdot),\cdot)$$

is $(\mathscr{F}\otimes\mathscr{B}\otimes\mathscr{B}(U))$-measurable for each $n\in\mathbb{Z}_{>0}$.

Property (I2) follows from (G2), (2.12) and (2.11), again by induction on $n\in\mathbb{Z}_{\geq 0}$. Indeed, at $n=0$, $\varphi(0,\omega,\cdot,u)$ is continuous for every $\omega\in\Omega$ and every $u\in\mathscr{U}$. So once (I2) has been proved for a certain value of $n\in\mathbb{Z}_{\geq 0}$, we conclude that

$$\varphi(n+1,\omega,\cdot,u) = f(\theta_n\omega,\varphi(n,\omega,\cdot,u),u_n(\omega))$$

is continuous for any $\omega\in\Omega$ and any $u\in\mathscr{U}$ as well.

Property (I3) follows from (2.12).

Before proving (I4) we first prove (I5) by induction on $n\in\mathbb{Z}_{\geq 0}$. Fix $\omega\in\Omega$, $x\in X$. Equation (2.12) gives us the base of the induction. Now assume (I5) holds for a certain value of $n\in\mathbb{Z}_{\geq 0}$. If $u,v\in\mathscr{U}$ are such that $u_j(\omega)=v_j(\omega)$ for $j=0,1,\ldots,n$, then $\varphi(n,\omega,x,u)=\varphi(n,\omega,x,v)$ by the induction hypothesis. So it follows from (2.11) that

$$\begin{aligned}\varphi(n+1,\omega,x,u) &= f(\theta_n\omega,\varphi(n,\omega,x,u),u_n(\omega))\\ &= f(\theta_n\omega,\varphi(n,\omega,x,v),v_n(\omega))\\ &= \varphi(n+1,\omega,x,v).\end{aligned}$$

This proves (I5).

It remains to prove (I4). For each arbitrarily fixed $p \in \mathbb{Z}_{\geq 0}$, we use induction on $n \in \mathbb{Z}_{\geq 0}$. For $n = 0$, (I4) holds in virtue of (I3) and (I5). For any $\omega \in \Omega$, we have $u_j(\omega) = (u \diamondsuit_p v)_j(\omega)$ for $j = 0, \ldots, p-1$. Therefore

$$\varphi(0, \theta_p\omega, \varphi(p, \omega, x, u), v) = \varphi(p, \omega, x, u) = \varphi(0 + p, \omega, x, u \diamondsuit_p v),$$

for any $x \in X$. Now suppose (I4) holds for some $n \in \mathbb{Z}_{\geq 0}$. Given $\omega \in \Omega$ and $x \in X$, set $y := \varphi(n, \theta_p\omega, x, u)$. Then

$$\begin{aligned}\varphi(n+1, \theta_p\omega, y, v) &= f(\theta_n\theta_p\omega, \varphi(n, \theta_p\omega, y, v), v_n(\theta_p\omega)) \\ &= f(\theta_{n+p}\omega, \varphi(n+p, \omega, x, u \diamondsuit_p v), (u \diamondsuit_p v)_{n+p}(\omega)) \\ &= \varphi(n+p+1, \omega, x, u \diamondsuit_p v).\end{aligned}$$

This completes the proof that $(\theta, \varphi, \mathscr{U})$ is an RDSI. □

Observe that we did not need (I1) in order to prove the first half of the theorem. So we could have in principle dropped this axiom from the definition of an RDSI and an analogous result would still hold. We remind the reader that (I1) was nevertheless used in showing that RDSI's restricted to θ-stationary inputs are RDS's (see Lemma 2.3 below).

From the construction of the generator f of an RDSI $(\theta, \varphi, \mathscr{U})$, it is clear how the dependence of the flow φ at time $n \in \mathbb{Z}_{\geq 0}$ and subject to $\omega \in \Omega$ on the input u is really through the value $u_n(\omega)$ of the input u. So when one shifts the Ω-argument ω of φ in the pullback trajectory to $\theta_{-n}\omega$, there is no need to change the input, since $\varphi(n, \theta_{-n}\omega, x(\theta_{-n}\omega), u)$ depends on $u_n(\theta_{-n}\omega)$ already. This is our second reason for defining the pullback trajectories of systems with inputs like so.

We now discuss the third and most important reason this is the mathematically sensible way of defining pullback trajectories for RDSI's. Let (θ, ψ) be a discrete RDS evolving on the state space $Z = X_1 \times X_2$:

$$\psi: \mathbb{Z}_{\geq 0} \times \Omega \times (X_1 \times X_2) \longrightarrow (X_1 \times X_2).$$

Let $g: \Omega \times Z \to Z$ be the generator of (θ, ψ). Suppose g can be written as

$$g(\omega, (x_1, x_2)) \equiv \begin{pmatrix} f_1(\omega, x_1) \\ f_2(\omega, x_2, h_1(\omega, x_1)) \end{pmatrix}, \tag{2.14}$$

where $f_1: \Omega \times X_1 \to X_1$ is the generator of some RDSO (θ, φ_1, h_1) with output space Y_1, and $f_2: \Omega \times X_2 \times U_2 \to X_2$ is the generator of some RDSI $(\theta, \varphi_2, \mathscr{U}_2)$ with input space $U_2 = Y_1$. Let $\pi_2: X_1 \times X_2 \to X_2$ be the projection onto the second coordinate. We use η_1 to denote the output trajectories of (θ, φ_1, h_1), ξ for the state trajectories of ψ, and ξ_2 for the state trajectories of $(\theta, \varphi_2, \mathscr{U}_2)$.

Theorem 2.2 (Projection of Pullback Equals Pullback of Projection). *For any random initial state*

$$z = (x_1, x_2) \in Z^{\Omega}_{\mathscr{B}(Z)} = (X_1)^{\Omega}_{\mathscr{B}(X_1)} \times (X_2)^{\Omega}_{\mathscr{B}(X_2)},$$

the following two identities hold:

(1) $\psi(n, \omega, z(\omega)) \equiv \begin{pmatrix} \varphi_1(n, \omega, x_1(\omega)) \\ \varphi_2(n, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}$, *and*

(2) $\pi_2(\check{\xi}^z_n(\omega)) \equiv (\check{\xi}_2)^{x_2,(\eta_1)^{x_1}}_n(\omega)$.

Proof. (1) For each fixed $\omega \in \Omega$ and $z \in Z^{\Omega}_{\mathscr{B}(Z)}$, we use induction on $n \in \mathbb{Z}_{\geq 0}$. At $n = 0$ we have

$$\psi(0, \omega, z(\omega)) = z(\omega) = \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \end{pmatrix} = \begin{pmatrix} \varphi_1(0, \omega, x_1(\omega)) \\ \varphi_2(0, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}.$$

Now suppose that (1) holds for some $n \in \mathbb{Z}_{\geq 0}$. Since

$$h_1(\theta_n\omega, \varphi_1(n, \omega, x_1(\omega))) = (\eta_1)^{x_1}_n(\omega)$$

by definition, it follows that

$$\begin{aligned} \psi(n+1, \omega, z(\omega)) &= g(\theta_n\omega, \psi(n, \omega, z(\omega))) \\ &= \begin{pmatrix} f_1(\theta_n\omega, \varphi_1(n, \omega, x_1(\omega))) \\ f_2(\theta_n\omega, \varphi_2(n, \omega, x_2(\omega), (\eta_1)^{x_1}), (\eta_1)^{x_1}_n(\omega)) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(n+1, \omega, x_1(\omega)) \\ \varphi_2(n+1, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}. \end{aligned}$$

This completes the induction.

(2) We prove by induction that (2) holds, for each $n \in \mathbb{Z}_{\geq 0}$, for all random initial states $z = (x_1, x_2) \in Z^{\Omega}_{\mathscr{B}(Z)}$, and all $\omega \in \Omega$. At $n = 0$ we have

$$\begin{aligned} \pi_2(\check{\xi}^z_0(\omega)) &= \pi_2(\psi(0, \omega, (x_1(\omega), x_2(\omega)))) \\ &= x_2(\omega) \\ &= \varphi_2(0, \omega, x_2(\omega), (\eta_1)^{x_1}) \\ &= (\check{\xi}_2)^{x_2,(\eta_1)^{x_1}}_0. \end{aligned}$$

Now assume (2) has been proved to hold for all integer values of n up to some $n_0 \geq 0$, for all random initial states $z = (x_1, x_2) \in Z^{\Omega}_{\mathscr{B}(Z)}$ and all $\omega \in \Omega$. Given $z = (x_1, x_2) \in Z^{\Omega}_{\mathscr{B}(Z)}$, define $\hat{z} = (\hat{x}_1, \hat{x}_2) \in Z^{\Omega}_{\mathscr{B}(Z)}$ by

$$\hat{z}(\omega) = g(\theta_{-1}\omega, z(\theta_{-1}\omega)) := \begin{pmatrix} f_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega)) \\ f_2(\theta_{-1}\omega, x_2(\theta_{-1}\omega), h_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega))) \end{pmatrix}, \quad \omega \in \Omega. \tag{2.15}$$

We have $(\eta_1)^{\hat{x}_1} = \rho_1((\eta_1)^{x_1})$ by Lemma 2.2 below, and also

$$h_1(\theta_{-(n_0+1)}\omega, x_1(\theta_{-(n_0+1)}\omega)) = (\eta_1)_0^{x_1}(\theta_{-(n_0+1)}\omega), \quad \omega \in \Omega.$$

Fix $\omega \in \Omega$ arbitrarily and denote $\hat{\omega} := \theta_{-(n_0+1)}\omega$. Then

$$\begin{aligned} \pi_2(\check{\xi}^z_{n_0+1}(\omega)) &= \pi_2(\psi(n_0+1, \hat{\omega}, z(\hat{\omega}))) \\ &= \pi_2(\psi(n_0, \theta_{-n_0}\omega, \psi(1, \hat{\omega}, z(\hat{\omega})))) \\ &= \pi_2(\psi(n_0, \theta_{-n_0}\omega, g(\hat{\omega}, z(\hat{\omega})))) \\ &= \pi_2(\psi(n_0, \theta_{-n_0}\omega, \hat{z}(\theta_{-n_0}\omega))) \\ &= \pi_2(\check{\xi}^{\hat{z}}_{n_0}(\omega)) \\ &= (\check{\xi}_2)^{\hat{x}_2, (\eta_1)^{\hat{x}_1}}_{n_0}(\omega) \end{aligned}$$

by the induction hypothesis. Now

$$\begin{aligned} (\check{\xi}_2)^{\hat{x}_2, (\eta_1)^{\hat{x}_1}}_{n_0}(\omega) &= \varphi_2(n_0, \theta_{-n_0}\omega, \hat{x}_2(\theta_{-n_0}\omega), (\eta_1)^{\hat{x}_1}) \\ &= \varphi_2(n_0, \theta_{-n_0}\omega, f_2(\hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{x_1}_0(\hat{\omega})), (\eta_1)^{\hat{x}_1}) \\ &= \varphi_2(n_0, \theta_{-n_0}\omega, \varphi_2(1, \hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{x_1}), \rho_1((\eta_1)^{\hat{x}_1})) \\ &= \varphi_2(n_0+1, \theta_{-(n_0+1)}\omega, x_2(\theta_{-(n_0+1)}\omega), (\eta_1)^{x_1}) \\ &= (\check{\xi}_2)^{x_2, (\eta_1)^{x_1}}_{n_0+1}(\omega). \end{aligned}$$

So

$$\pi_2(\check{\xi}^z_{n_0+1}(\omega)) = (\check{\xi}_2)^{x_2, (\eta_1)^{x_1}}_{n_0+1}(\omega).$$

Since $z = (x_1, x_2) \in Z^{\Omega}_{\mathscr{B}(Z)}$ and $\omega \in \Omega$ were arbitrary, this completes the inductive step. □

The left hand side of (2) in the proposition above is the projection over the second coordinate of the pullback trajectory starting at $z = (x_1, x_2)$ of the RDS (θ, ψ). The right hand side is the pullback trajectory of the RDSI $(\theta, \varphi_2, \mathscr{U}_2)$ starting at x_2 and subject to the input $(\eta_1)^{x_1}$, the output trajectory of (θ, φ_1, h_1) starting at x_1. Theorem 2.2 then says that they coincide. An analogous result holds in continuous time for systems generated by random differential equations. These provide the motivation for the definition of cascades of systems with inputs and outputs, an introductory discussion of which is carried out in Sect. 2.4.2.

We now state and prove the technical lemma referred to in the proof of item (2) in Theorem 2.2:

Lemma 2.2. *Let $f: \Omega \times X \to X$ be the generator of a discrete RDSO (θ, φ, h). Given $x \in X^{\Omega}_{\mathscr{B}}$, let $\hat{x} \in X^{\Omega}_{\mathscr{B}}$ be defined by*

$$\hat{x}(\omega) := f(\theta_{-1}\omega, x(\theta_{-1}\omega)), \quad \omega \in \Omega.$$

Then $\eta^{\hat{x}} = \rho_1(\eta^x)$.

Proof. Indeed, we have

$$\begin{aligned}
\eta^{\hat{x}}_n(\omega) &= h(\theta_n\omega, \varphi(n, \omega, \hat{x}(\omega))) \\
&= h(\theta_n\omega, \varphi(n, \omega, f(\theta_{-1}\omega, x(\theta_{-1}\omega)))) \\
&= h(\theta_n\omega, \varphi(n, \omega, \varphi(1, \theta_{-1}\omega, x(\theta_{-1}\omega)))) \\
&= h(\theta_{n+1}\theta_{-1}\omega, \varphi(n+1, \theta_{-1}\omega, x(\theta_{-1}\omega))) \\
&= \eta^x_{n+1}(\theta_{-1}\omega) \\
&= (\rho_1(\eta^x))_n(\omega),
\end{aligned}$$

for every $n \in \mathbb{Z}_{\geq 0}$ and every $\omega \in \Omega$. □

2.3.2 *θ*-Stationary Inputs

The concept of RDSI subsumes that of an RDS, as we shall see below. Denote the subset of $\mathscr{S}^U_\theta$ consisting of θ-stationary inputs by $\bar{\mathscr{S}}^U_\theta$. We identify $\bar{\mathscr{S}}^U_\theta$ and $U^{\Omega}_{\mathscr{B}}$ via Lemma 2.1.

Let $(\theta, \varphi, \mathscr{U})$ be a RDSI, and suppose that $\bar{u} \in \mathscr{U} \cap \bar{\mathscr{S}}^U_\theta$ is some θ-stationary input. Consistent with the convention that an overbar is used to indicate the θ-stationary process associated with a given random variable, we remove the bar to denote the random variable associated with a given θ-stationary process. So we denote by u the random variable in $U^{\Omega}_{\mathscr{B}}$ associated via Lemma 2.1 with $\bar{u}$. We then define

$$\varphi_u := \varphi(\cdot, \cdot, \cdot, \bar{u}) \colon \mathscr{T}_{\geq 0} \times \Omega \times X \longrightarrow X.$$

Lemma 2.3. *φ_u is a crude cocycle.*

Proof. It follows from condition (I1) and [12, p. 65, Proposition 2.34] that φ_u is measurable. From (I2), $\varphi_u(t, \omega, \cdot)$ is continuous for each $(t, \omega) \in \mathscr{T}_{\geq 0} \times \Omega$, yielding (S1). From (I3), we know that $\varphi_u(0, \omega, \cdot) = \mathrm{id}_X$ for every $\omega \in \Omega$. So to verify (S2′) it remains to prove that φ_u satisfies the "crude cocycle property." Let $\tilde{\Omega} \subseteq \Omega$ be a θ-invariant subset of full measure such that

$$(\rho_s(\bar{u}))_t(\omega) = \bar{u}_t(\omega), \quad \forall s, t \in \mathscr{T}_{\geq 0}, \ \forall \omega \in \tilde{\Omega}. \tag{2.16}$$

Fix arbitrarily $\omega \in \tilde{\Omega}$. For any $s, t \in \mathscr{T}_{\geq 0}$, we have $\theta_s\omega \in \tilde{\Omega}$ by θ-invariance, and so it follows from (2.16) and (I5) that

$$\varphi(t, \theta_s\omega, \varphi_u(s, \omega, x), \rho_s(\bar{u})) = \varphi(t, \theta_s\omega, \varphi_u(s, \omega, x), \bar{u}).$$

It then follows from (I4)—see Remark 2.1—that

$$\begin{aligned}
\varphi_u(t + s, \omega, x) &= \varphi(t + s, \omega, x, \bar{u}) \\
&= \varphi(t, \theta_s\omega, \varphi(s, \omega, x, \bar{u}), \rho_s(\bar{u})) \\
&= \varphi(t, \theta_s\omega, \varphi_u(s, \omega, x), \bar{u}) \\
&= \varphi_u(t, \theta_s\omega, \varphi_u(s, \omega, x)).
\end{aligned}$$

So (S2′) is satisfied with $\Omega_s := \tilde{\Omega}$ for every $s \in \mathscr{T}_{\geq 0}$. □

Proposition 2.3. *If X is a locally compact and locally connected, Hausdorff topological space, then φ_u can be perfected.*

Proof. This follows straight from Proposition 2.1. □

Note that, since $\tilde{\Omega}$ in the proof of Lemma 2.3 is θ-invariant, so is its complement in Ω, namely $\Omega \backslash \tilde{\Omega}$. So Proposition 2.3 could have also been proved directly by redefining φ_u to take an arbitrarily fixed value of $x_0 \in X$ on the set

$$\mathscr{T}_{\geq 0} \times (\Omega \backslash \tilde{\Omega}) \times X.$$

Whenever the state space X is such that φ_u can be perfected, we shall assume that φ_u has already been replaced by an indistinguishable perfection and then refer to the resulting RDS (θ, φ_u).

2.3.3 Tempered Random Sets

Recall that, given a topological space X, a multifunction $D\colon \Omega \to 2^X$ is said to be a *random set* if

$$D^{-1}(U) := \{\omega \in \Omega;\ D(\omega) \cap U \neq \varnothing\} \in \mathscr{F}$$

for every open set $U \subseteq X$ (see [16, Chap. 2]). In this work, we shall be concerned exclusively with so-called *Polish spaces*; that is, separable topological spaces generated by a metric with respect to which they are complete. In such spaces, the definition above is known [16, p. 142, Proposition 1.4] to be equivalent to the requirement that

$$\omega \longmapsto \operatorname{dist}(x, D(\omega)) := \inf_{y \in D(\omega)} d(x, y), \quad \omega \in \Omega,$$

defines a Borel-measurable[7] map $\Omega \to \bar{\mathbb{R}}_{\geq 0}$ for each $x \in X$.

Definition 2.7 (Tempered Random Variables). A nonnegative, Borel-measurable function $r\colon \Omega \to \mathbb{R}_{\geq 0}$ is said to be a *tempered random variable* (*with respect to the underlying MPDS* θ) if, for every $\gamma > 0$,

$$\sup_{s \in \mathscr{T}} r(\theta_s \omega)\, \mathrm{e}^{-\gamma|s|} < \infty, \quad \tilde{\forall} \omega \in \Omega.$$

We denote the family of nonnegative, tempered (with respect to θ) random variables $\Omega \to \mathbb{R}_{\geq 0}$ by $(\mathbb{R}_{\geq 0})^{\Omega}_{\theta}$.

Observe that we do not require the bound to be independent of $\omega \in \Omega$. In fact, if it were, then r would have been essentially bounded. More precisely, suppose that, for some $\gamma > 0$, there exists a $K_\gamma \geq 0$ such that

$$\sup_{s \in \mathscr{T}} r(\theta_s \omega)\, \mathrm{e}^{-\gamma|s|} \leq K_\gamma, \quad \tilde{\forall} \omega \in \Omega.$$

Then

$$0 \leq r(\omega) \leq \sup_{s \in \mathscr{T}} r(\theta_s \omega)\, \mathrm{e}^{-\gamma|s|} \leq K_\gamma, \quad \tilde{\forall} \omega \in \Omega.$$

So r is actually essentially bounded.

Definition 2.8 (Tempered Random Set). Let (X, d) be a metric space. A random set $D\colon \Omega \to 2^X$ is said to be *tempered* (*with respect to* θ) if there exist $x_0 \in X$ and a nonnegative tempered random variable $r\colon \Omega \to \mathbb{R}_{\geq 0}$ such that

$$D(\omega) \subseteq \{x \in X;\ d(x, x_0) \leq r(\omega)\}, \quad \forall \omega \in \Omega. \tag{2.17}$$

A Borel-measurable map $v\colon \Omega \to X$ is said to be a *tempered random variable* (*with respect to* θ) if the random singleton defined by $\omega \mapsto \{v(\omega)\}$, $\omega \in \Omega$, is a tempered random set.

We denote the family of tempered (with respect to θ) random sets $\Omega \to 2^X$ by $(2^X)^{\Omega}_{\theta}$. Likewise, the family of tempered (with respect to θ) random variables $\Omega \to X$ is denoted by X^{Ω}_{θ}.

Lemma 2.4. *Suppose θ is an MPDS, $(X, \|\cdot\|)$ is a normed space over $\mathbb{R}$, and let $R_1, R_2 \in X^{\Omega}_{\theta}$, $r \in \mathbb{R}^{\Omega}_{\theta}$, and $c \in \mathbb{R}$. Then*

[7] Our convention is that $\inf \varnothing := +\infty$.

(1) $R_1 + R_2$ *is tempered.*
(2) cR_1 *is tempered.*
(3) rR_1 *is tempered; in particular, the product of two real-valued tempered random variables is tempered.*

Proof. (1) Indeed, for any $\gamma > 0$ and any $\omega \in \tilde{\Omega}$, we have

$$\sup_{s\in\mathscr{T}} \|(R_1 + R_2)(\theta_s\omega)\|e^{-\gamma|s|} \leq \sup_{s\in\mathscr{T}} \|R_1(\theta_s\omega)\|e^{-\gamma|s|} + \sup_{s\in\mathscr{T}} \|R_2(\theta_s\omega)\|e^{-\gamma|s|} < \infty,$$

where we write $(R_1 + R_2)(\theta_s\omega)$ for $R_1(\theta_s\omega) + R_2(\theta_s\omega)$. So both $R_1 + R_2$ is tempered.

(2) follows from (3), which we now prove. Given $\gamma > 0$ and $\omega \in \tilde{\Omega}$, apply the definition of tempered random variable for $\gamma/2$:

$$\begin{aligned}\sup_{s\in\mathscr{T}} \|r(\theta_s\omega)R_1(\theta_s\omega)\|e^{-\gamma|s|} &= \sup_{s\in\mathscr{T}} |r(\theta_s\omega)|e^{-\frac{\gamma}{2}|s|}\|R_1(\theta_s\omega)\|e^{-\frac{\gamma}{2}|s|} \\ &\leq \left(\sup_{s\in\mathscr{T}} |r(\theta_s\omega)|e^{-\frac{\gamma}{2}|s|}\right)\left(\sup_{s\in\mathscr{T}} \|R_1(\theta_s\omega)\|e^{-\frac{\gamma}{2}|s|}\right) \\ &< \infty.\end{aligned}$$

Thus rR_1 is tempered. □

In other words, X_θ^Ω is a real vector space, and also a module over the ring of real-valued tempered random variables.

We now introduce concepts of convergence and continuity taking into account the notion of temperedness just introduced.

Definition 2.9 (Tempered Convergence). Suppose θ is an MPDS and (X, d) is a metric space. We say that a net $(\xi_\alpha)_{\alpha\in A}$ in $X_{\mathscr{B}}^\Omega$ converges in the *tempered sense* to a random variable $\xi_\infty \in X_{\mathscr{B}}^\Omega$ if there exists a nonnegative, tempered random variable $r\colon \Omega \to \mathbb{R}_{\geq 0}$ and an $\alpha_0 \in A$ such that

(1) $\xi_\alpha(\omega) \to \xi_\infty(\omega)$ as $\alpha \to \infty$ for θ-almost all $\omega \in \Omega$, and
(2) $d(\xi_\alpha(\omega), \xi_\infty(\omega)) \leq r(\omega)$ for all $\alpha \succcurlyeq \alpha_0$, for θ-almost all $\omega \in \Omega$.

In this case we denote $\xi_\alpha \to_\theta \xi_\infty$ (as $\alpha \to \infty$).

Definition 2.10 (Tempered Continuity). Suppose θ is an MPDS and X, U are metric spaces. A map $\mathscr{K}\colon \mathscr{U} \subseteq U_{\mathscr{B}}^\Omega \to X_{\mathscr{B}}^\Omega$ is said do be *tempered continuous* if $\mathscr{K}(u_\alpha) \to_\theta \mathscr{K}(u_\infty)$ for every net $(u_\alpha)_{\alpha\in A}$ in $\mathscr{U}$ such that $u_\alpha \to_\theta u_\infty$ for some $u_\infty \in \mathscr{U}$.

We close this subsection with the definition of several asymptotic behavior concepts. Let X be a metric space. Given $\xi \in \mathscr{S}_\theta^X$ and $\tau \geq 0$, we call the multifunction $\beta_\xi^\tau\colon \Omega \to 2^X \setminus \{\varnothing\}$, defined by

$$\beta_\xi^\tau(\omega) := \{\xi_t(\theta_{-t}\omega);\ t \geq \tau\}, \quad \omega \in \Omega,$$

the *tail* (*from moment* τ) of the pullback trajectories of ξ. If a θ-stochastic process $\xi \in \mathscr{S}_\theta^X$ is such that there exists a $\tau_\xi \geq 0$ such that $\beta_\xi^\tau(\omega)$ is precompact for all $\tau \geq \tau_\xi$, for θ-almost all $\omega \in \Omega$, then we say that ξ is *eventually precompact*. We denote the subset of all eventually precompact θ-stochastic processes $\xi \in \mathscr{S}_\theta^X$ by $\mathscr{K}_\theta^X$. A θ-stochastic process $\xi \in \mathscr{S}_\theta^X$ is said to be *tempered* if there exists a tempered random set $D \in (2^X)_\theta^\Omega$ such that

$$\beta_\xi^\tau(\omega) \subseteq D(\omega), \quad \forall \tau \geq 0,\ \tilde{\forall}\omega \in \Omega; \tag{2.18}$$

in other words,

$$\xi_t(\theta_{-t}\omega) = \check{\xi}_t(\omega) \in D(\omega), \quad \forall t \geq 0,\ \tilde{\forall}\omega \in \Omega. \tag{2.19}$$

Any $D \in (2^X)_\theta^\Omega$ for which the relation above holds is called a *rest set*. The subset of $\mathscr{S}_\theta^X$ consisting of all tempered θ-stochastic processes $\xi \in \mathscr{S}_\theta^X$ is denoted by $\mathscr{V}_\theta^X$. Observe that, in virtue of θ-invariance, condition (2.19) is equivalent to

$$\xi_t(\omega) \in D(\theta_t\omega), \quad \forall t \geq 0,\ \tilde{\forall}\omega \in \Omega.$$

We further motivate the concept of temperedness just introduced. The idea is to have a term to talk about θ-stochastic processes which, as far as their oscillatory behavior is concerned, look somewhat like a θ-stationary process generated by a tempered random variable. Since this pertains to long-term behavior, this property should be preserved by shifting or concatenating tempered stochastic processes. Indeed, it is not difficult to show that (1) θ-stationary processes generated by tempered random variables are tempered, (2) $\rho_s(u)$ is tempered for any tempered u, and (3) $u\Diamond_s v$ is tempered for any tempered u, v.

Definition 2.11 (Tempered RDSI). An RDSI $(\theta, \varphi, \mathscr{U})$ is said to be *tempered* if the trajectories $\xi^{x,u}$ are tempered for every tempered initial state $x \in X_\theta^\Omega$ and every tempered input $u \in \mathscr{U}$.

2.3.4 Input to State Characteristics

Let $(\theta, \varphi, \mathscr{U})$ be an RDSI and suppose that $\bar{u} \in \mathscr{U}$ is a θ-stationary process, with generating random variable u (refer to Lemma 2.1). Any equilibrium ξ of the RDS (θ, φ_u) will be referred to as an *equilibrium associated to* $\bar{u}$ (or *to* u). The set of all equilibria associated to $\bar{u}$ (or to u) is denoted as $\mathscr{E}(\bar{u})$ (we may also write $\mathscr{E}(u)$). So an element $\xi \in \mathscr{E}(\bar{u})$ is a random variable $\Omega \to X$ such that

$$\varphi_u(t, \theta_{-t}\omega, \xi(\theta_{-t}\omega)) = \xi(\omega), \quad \forall t \geq 0, \quad \tilde{\forall}\omega \in \Omega. \tag{2.20}$$

When we have a "proper" RDS (θ, φ), we write simply $\mathscr{E}$ for the set of equilibria of (θ, φ).

For deterministic systems—when Ω is a singleton and we may identify the set of θ-inputs $\mathscr{U}$ with the input space U—, if the set $\mathscr{E}(\bar{u})$ consists of a single, globally attracting equilibrium, then the mapping $u \mapsto \mathscr{E}(\bar{u})$, $u \in U$, is the object called the "input to state characteristic" in the literature on monotone i/o systems. For systems with outputs, composition with the output map h provides the "input to output" characteristic [1]. One of the contributions of this work is the extension of these concepts to RDSI's and RDSIO's.

In this section we introduce the notion of input to state characteristics for RDSI's and discuss a class of examples. Systems with outputs will be considered in greater detail in the next section. For reasons which will be illustrated in Example 2.3 and become clearer in the proof of Theorem 2.3 (CICS), further conditions on the convergence of the states are needed.

Definition 2.12 (I/S Characteristic). An RDSI $(\theta, \varphi, \mathscr{U})$ is said to have an *input to state* (*i/s*) *characteristic* $\mathscr{K} : U_\theta^\Omega \to X_\theta^\Omega$ if

$$U_\theta^\Omega \subseteq \mathscr{U}$$

and

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathscr{K}(u) \quad \text{as} \quad t \to \infty,$$

for every $x \in X_\theta^\Omega$, for every $u \in U_\theta^\Omega$.

Example 2.3 below illustrates the concepts of tempered RDSI (Definition 2.11) and i/s characteristics (Definition 2.12 above). Temperedness features in said example will be a special case (with $p = 1$ or $p = \infty$) of the general result below.

Proposition 2.4. *Suppose $r : \Omega \to \mathbb{R}_{\geq 0}$ is a tempered random variable. For each $\gamma > 0$ and each $p \in [1, \infty]$, the map*

$$\omega \longmapsto \|r(\theta.\omega)\, \mathrm{e}^{-\gamma|\cdot|}\|_{L^p(\mathbb{R})}, \quad \omega \in \Omega,$$

is a tempered random variable. Moreover, temperedness bounds are uniform in $p \in [1, \infty]$; that is, for each $\gamma > 0$ and each $\delta > 0$,

$$\sup_{p \in [1,\infty]} \sup_{s \in \mathbb{R}} \|r(\theta.\theta_s\omega)\, \mathrm{e}^{-\gamma|\cdot|}\|_{L^p(\mathbb{R})}\, \mathrm{e}^{-\delta|s|} < \infty, \quad \tilde{\forall}\omega \in \Omega.$$

Proof. For each $\mu > 0$, set

$$K_{\mu,\omega} := \sup_{s \in \mathbb{R}} r(\theta_s\omega)\, \mathrm{e}^{-\mu|s|}$$

for every $\omega \in \Omega$ such that the supremum above is finite. Since r is tempered by assumption, this will be true for θ-almost all $\omega \in \Omega$.

Fix arbitrarily $\gamma > 0$ and choose any $\delta > 0$. We consider two different cases.

(Case $1 \le p < \infty$) Setting $m := \min\{\gamma, \delta\} > 0$ and using the triangle inequality we obtain

$$\begin{aligned}\|r(\theta.\theta_s\omega)\,\mathrm{e}^{\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} &= \left(\int_{-\infty}^{\infty} |r(\theta_{t+s}\omega)\,\mathrm{e}^{-\gamma|t|-\delta|s|}\,|^p\,dt\right)^{1/p} \\ &\le \left(\int_{-\infty}^{\infty} |r(\theta_{t+s}\omega)\,\mathrm{e}^{-m|t+s|}\,|^p\,dt\right)^{1/p} \\ &\le K_{\frac{m}{2},\omega}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{pm}{2}|t+s|}\,dtsw\right)^{1/p} \\ &= K_{\frac{m}{2},\omega}\left(\tfrac{4}{pm}\right)^{1/p},\end{aligned}$$

which is finite for all $s \in \mathbb{R}$, for θ-almost all $\omega \in \Omega$. In fact, since the map

$$p \longmapsto K_{\frac{m}{2},\omega}\left(\frac{4}{pm}\right)^{1/p}, \quad 1 \le p < \infty, \tag{2.21}$$

is continuous in p and

$$\lim_{p\to\infty} K_{\frac{m}{2},\omega}\left(\frac{4}{pm}\right)^{1/p} = 1,$$

we then know that the map in (2.21) is bounded. Thus

$$M_{\gamma,\delta,\omega} := \sup_{p\in[1,\infty)}\ \sup_{s\in\mathbb{R}} \|r(\theta.\theta_s\omega)\,\mathrm{e}^{-\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} < \infty, \quad \tilde{\forall}\omega \in \Omega.$$

(Case $p = \infty$) The trick is basically the same as before. We have

$$\begin{aligned}\|r(\theta.\theta_s\omega)\,\mathrm{e}^{\gamma|\cdot|}\,\|_{L^\infty(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} &= \sup_{t\in\mathbb{R}} r(\theta_{t+s}\omega)\,\mathrm{e}^{-\gamma|t|-\delta|s|} \\ &\le \sup_{t\in\mathbb{R}} r(\theta_{t+s}\omega)\,\mathrm{e}^{-m|t+s|} \\ &= K_{m,\omega},\end{aligned}$$

which is finite for all $s \in \mathbb{R}$, for θ-almost all $\omega \in \Omega$.

Combining both cases we conclude that

$$\sup_{p\in[1,\infty]} \sup_{s\in\mathbb{R}} \|r(\theta.\theta_s\omega)\,\mathrm{e}^{-\gamma|\cdot|}\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} = \max\{M_{\gamma,\delta,\omega}, K_{m,\omega}\},$$

which is finite for θ-almost all $\omega \in \Omega$. Since $\gamma, \delta > 0$ were chosen arbitrarily, this completes the proof. □

Example 2.3 (I/S Characteristics for RDSI's Generated by Linear RDEI's). Consider the RDSI $(\theta, \varphi, \mathscr{S}_\infty^U)$ from Example 2.2, generated by the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega), \quad t \geq 0, \quad u \in \mathscr{S}_\infty^U, \tag{2.22}$$

where $X = \mathbb{R}^n$, $U = \mathbb{R}^k$, and $A\colon \Omega \to \mathbb{R}^{n\times n}$ and $B\colon \Omega \to \mathbb{R}^{n\times k}$ are random matrices such that

$$t \longmapsto A(\theta_t\omega), \quad t \geq 0, \quad \text{and} \quad t \longmapsto B(\theta_t\omega), \quad t \geq 0,$$

are locally essentially bounded for every $\omega \in \Omega$. Now suppose in addition that A, B are such that

(L1) B is tempered and
(L2) there exist a $\lambda > 0$ and a nonnegative, tempered random variable $\gamma \in (\mathbb{R}_\geq)_\theta^\Omega$ such that the fundamental matrix solution Ξ of the homogeneous part of (2.22) satisfies

$$\|\Xi(s, s+r, \omega)\| \leq \gamma(\theta_s\omega)\,\mathrm{e}^{-\lambda r}, \quad \forall s \in \mathbb{R}, \quad \forall r \geq 0, \quad \tilde{\forall}\omega \in \Omega.$$

Then $(\theta, \varphi, \mathscr{S}_\infty^U)$ is tempered (in the sense of Definition 2.11) and has a continuous input to state characteristic $\mathscr{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ (refer to Definition 2.12). We will prove this in several steps, indicated below.

Construction of $\mathscr{K}\colon U_\theta^\Omega \to X_\theta^\Omega$. We first claim that the limit

$$\lim_{t\to\infty} \check{\xi}_t^{x,\bar{u}}(\omega) = \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma \tag{2.23}$$

exists for each $x \in X_\theta^\Omega$ and each $u \in U_\theta^\Omega$, for θ-almost $\omega \in \Omega$. Let Φ and Ψ be as in Example 2.2, so that we may write

$$\varphi(t, \omega, x, u) \equiv \Phi(t, \omega, x) + \Psi(t, \omega, u).$$

So it is enough to show that

$$\lim_{t\to\infty} \Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = 0, \quad \forall x \in X_\theta^\Omega, \quad \tilde{\forall}\omega \in \Omega, \tag{2.24}$$

and that

$$\lim_{t\to\infty} \Psi(t,\theta_{-t}\omega,\bar{u}) = \int_{-\infty}^{0} \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma, \quad \forall u \in U_\theta^\Omega, \quad \tilde{\forall}\omega \in \Omega. \tag{2.25}$$

Fix arbitrarily $x \in X_\theta^\Omega$ and let $\omega \in \Omega$ be such that

$$K_{\omega,\frac{\lambda}{2},x} := \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega)|x(\theta_s\omega)|\,\mathrm{e}^{-\frac{\lambda}{2}|s|} < \infty, \tag{2.26}$$

where $\lambda > 0$ and γ nonnegative and tempered are given by (L2). Combining (L2) and (2.26), we obtain

$$\begin{aligned} |\Phi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))| &= |\Xi(0,t,\theta_{-t}\omega)\cdot x(\theta_{-t}\omega)| \\ &\le \gamma(\theta_{-t}\omega)\,\mathrm{e}^{-\lambda t}\,|x(\theta_{-t}\omega)| \\ &= \left(\gamma(\theta_{-t}\omega)|x(\theta_{-t}\omega)|\,\mathrm{e}^{-\frac{\lambda}{2}|-t|}\right)\mathrm{e}^{-\frac{\lambda}{2}t} \\ &\le K_{\omega,\frac{\lambda}{2},x}\,\mathrm{e}^{-\frac{\lambda}{2}t}, \quad \forall t \ge 0. \end{aligned}$$

Hence

$$|\Phi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))| \longrightarrow 0 \quad \text{as} \quad t\to\infty.$$

Since $K_{\omega,\frac{\lambda}{2},x}$ is finite for θ-almost all $\omega \in \Omega$—recall that, by Lemma 2.4(3), the product of two tempered random variables is tempered—, this holds θ-almost everywhere. So since $x \in X_\theta^\Omega$ was chosen arbitrarily, this proves (2.24).

Now fix arbitrarily $u \in U_\theta^\Omega$. Then by (F2) and a change of variables,

$$\begin{aligned} \Psi(t,\theta_{-t}\omega,\bar{u}) &= \int_0^t \Xi(\sigma,t,\theta_{-t}\omega)B(\theta_{\sigma-t}\omega)u(\theta_{\sigma-t}\omega)\,d\sigma \\ &= \int_0^t \Xi(\sigma-t,0,\omega)B(\theta_{\sigma-t}\omega)u(\theta_{\sigma-t}\omega)\,d\sigma \\ &= \int_{-t}^0 \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma, \quad \forall (t,\omega) \in \mathbb{R}_{\ge 0}\times\Omega. \end{aligned}$$

In virtue of (L2), for each $\omega \in \Omega$ such that

$$L_{\omega,\frac{\lambda}{2},u} := \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega)\|B(\theta_s\omega)\|\cdot|u(\theta_s\omega)|\,\mathrm{e}^{-\frac{\lambda}{2}|s|} < \infty, \tag{2.27}$$

we have

$$\begin{aligned} |\Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)| &\le \gamma(\theta_\sigma\omega)\,\mathrm{e}^{-\lambda|\sigma|}\,\|B(\theta_\sigma\omega)\|\cdot|u(\theta_\sigma\omega)| \\ &\le L_{\omega,\frac{\lambda}{2},u}\,\mathrm{e}^{-\frac{\lambda}{2}|\sigma|}, \quad \forall \sigma \in \mathbb{R}. \end{aligned}$$

Since

$$\sigma \longmapsto L_{\omega,\frac{\lambda}{2},u}\, \mathrm{e}^{-\frac{\lambda}{2}|\sigma|}, \quad \sigma \in \mathbb{R},$$

is integrable on $(-\infty, \infty)$, so is

$$\sigma \longmapsto \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u(\theta_\sigma \omega), \quad \sigma \in \mathbb{R}.$$

In particular, it follows from dominated convergence that the limit

$$\begin{aligned} \lim_{t\to\infty} \Psi(t, \theta_{-t}\omega, \bar{u}) &= \lim_{t\to\infty} \int_{-t}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u(\theta_\sigma \omega)\, d\sigma \\ &= \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u(\theta_\sigma \omega)\, d\sigma \end{aligned}$$

exists. Finally, observe that, for each $u \in U_\theta^\Omega$, $L_{\omega,\frac{\lambda}{2},u}$ as defined in (2.27) is finite for θ-almost all $\omega \in \Omega$. This establishes (2.25). We have then proved that (2.23) holds for each $x \in X_\theta^\Omega$ and each $u \in U_\theta^\Omega$, for θ-almost all $\omega \in \Omega$.

Define $\mathscr{K}: U_\theta^\Omega \to X_\mathscr{B}^\Omega$ by

$$(\mathscr{K}(u))(\omega) := \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u(\theta_\sigma \omega)\, d\sigma, \quad \tilde{\forall}\omega \in \Omega.$$

It remains to show that $\mathscr{K}(U_\theta^\Omega) \subseteq X_\theta^\Omega$. Indeed, fix $u \in U_\theta^\Omega$ arbitrarily. It follows from the computations above that

$$\begin{aligned} |(\mathscr{K}(u))(\omega)| &\le \int_{-\infty}^{0} \gamma(\theta_\sigma\omega) \|B(\theta_\sigma \omega)\| \cdot |u(\theta_\sigma \omega)|\, \mathrm{e}^{-\lambda|\sigma|}\, d\sigma \\ &\le \int_{-\infty}^{\infty} \gamma(\theta_\sigma\omega) \|B(\theta_\sigma \omega)\| \cdot |u(\theta_\sigma \omega)|\, \mathrm{e}^{-\lambda|\sigma|}\, d\sigma \\ &= \|(\gamma \|B\| \cdot |u|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^1(\mathbb{R})}, \quad \tilde{\forall}\omega \in \Omega. \end{aligned}$$

From Proposition 2.4,

$$\omega \longmapsto \|(\gamma \|B\| \cdot |u|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^1(\mathbb{R})}, \quad \omega \in \Omega,$$

is tempered. Thus $\mathscr{K}(u)$ is also tempered.

$\mathscr{K}$ is an i/s characteristic. To show that $\mathscr{K}$ is an i/s characteristic, it remains to show that the convergence in both (2.24) and (2.25) is tempered.

Fix $x \in X_\theta^\Omega$ arbitrarily. From the estimates above, we have

$$\begin{aligned} |\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| &\le \gamma(\theta_{-t}\omega) |x(\theta_{-t}\omega)|\, \mathrm{e}^{-\lambda t} \\ &\le \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega) |x(\theta_s\omega)|\, \mathrm{e}^{-\lambda|s|} \\ &= \|(\gamma|x|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^\infty(\mathbb{R})}, \quad \forall t \ge 0, \quad \tilde{\forall}\omega \in \Omega. \end{aligned}$$

It follows from Proposition 2.4 (applied with $p = \infty$) that

$$\omega \longmapsto \|(\gamma|x|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\,\|_{L^\infty(\mathbb{R})}, \quad \omega \in \Omega,$$

is tempered. We conclude that the convergence in (2.24) is tempered.

Similarly, for any arbitrarily fixed $u \in U_\theta^\Omega$, we have

$$\begin{aligned}|\Psi(t, \theta_{-t}\omega, \bar{u}) - (\mathscr{K}(u))(\omega)| &= \left|\int_{-\infty}^{-t} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) \cdot u(\theta_\sigma \omega)\, d\sigma\right| \\ &\leq \int_{-\infty}^{\infty} |\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) \cdot u(\theta_\sigma \omega)|\, d\sigma \\ &= \|(\gamma \|B\| \cdot |u|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})}\end{aligned}$$

for all $t \geq 0$ and θ-almost all $\omega \in \Omega$. As we saw above, the rightmost term in these inequalities is a tempered random variable. So the convergence in (2.25) is also tempered.

$\mathscr{K}$ is continuous. Suppose that $u_\alpha \to_\theta u_\infty \in U_\theta^\Omega$ for some net $(u_\alpha)_{\alpha \in A}$ in U_θ^Ω. Let $\alpha_0 \in A$ and $r \in (\mathbb{R}_{\geq 0})_\theta^\Omega$ be such that

$$|u_\alpha(\omega) - u_\infty(\omega)| \leq r(\omega), \quad \forall \alpha \geq \alpha_0, \quad \tilde{\forall} \omega \in \Omega.$$

Then

$$\begin{aligned}&|(\mathscr{K}(u_\alpha))(\omega) - (\mathscr{K}(u_\infty))(\omega)| \\ &\quad = \left|\int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) \cdot (u_\alpha(\theta_\sigma \omega) - u_\infty(\theta_\sigma \omega))\, d\sigma\right| \\ &\quad \leq \int_{-\infty}^{\infty} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot r(\theta_\sigma \omega)\, d\sigma\end{aligned}$$

for every $\alpha \geq \alpha_0$, for θ-almost all $\omega \in \Omega$. As above, we can combine (L2), the temperedness of γ, B and r, Lemma 2.4(3) and Proposition 2.4 to conclude that

$$\omega \longmapsto \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot r(\theta_\sigma \omega), \quad \omega \in \Omega,$$

is integrable for θ-almost all $\omega \in \Omega$, and that the map

$$\omega \longmapsto \int_{-\infty}^{\infty} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot r(\theta_\sigma \omega)\, d\sigma, \quad \omega \in \Omega,$$

is tempered. In particular, since

$$|u_\alpha(\omega) - u_\infty(\omega)| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty, \quad \tilde{\forall} \omega \in \Omega,$$

it follows from dominated convergence that the map

$$|(\mathscr{K}(u_\alpha))(\omega) - (\mathscr{K}(u_\infty))(\omega)| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty, \qquad \tilde{\forall}\omega \in \Omega,$$

as well. This shows that $\mathscr{K}(u_\alpha) \to_\theta \mathscr{K}(u_\infty)$. Since $u_\infty \in U_\theta^\Omega$ and the net $(u_\alpha)_{\alpha\in A}$ converging to it were arbitrary, this shows $\mathscr{K}$ is continuous.

φ is tempered. The argument here goes along the same lines. Fix arbitrarily any tempered input $u \in \mathscr{S}_\infty^U$ and any tempered initial state $x \in X_\theta^\Omega$. When we were showing that $\mathscr{K}$ is an i/s characteristic above, we saw that

$$|\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| \le r_1(\omega), \quad \forall t \ge 0, \quad \tilde{\forall}\omega \in \Omega,$$

where $r_1\colon \Omega \to \mathbb{R}_{\ge 0}$ is a tempered random variable defined by

$$r_1(\omega) := \|(\gamma|x|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^\infty(\mathbb{R})}, \quad \omega \in \Omega.$$

Now let $D \in (2^U)_\theta^\Omega$ be a (tempered) rest set for u. Let $r \in (\mathbb{R}_{\ge 0})_\theta^\Omega$ be such that

$$D(\omega) \subseteq \{u \in U;\ \|u\| \le r(\omega)\}, \quad \forall \omega \in \Omega.$$

Thus indeed

$$\|u_t(\theta_{-t}\omega)\| \le r(\omega), \quad \forall t \ge 0, \quad \tilde{\forall}\omega \in \Omega.$$

Then

$$\begin{aligned}
|\Psi(t, \theta_{-t}\omega, u)| &= \left| \int_0^t \Xi(\sigma, t, \theta_{-t}\omega) B(\theta_{\sigma-t}\omega) u_\sigma(\theta_{-t}\omega)\, d\sigma \right| \\
&\le \int_0^t \|\Xi(\sigma, t, \theta_{-t}\omega) B(\theta_{\sigma-t}\omega)\| \cdot |u_\sigma(\theta_{-\sigma}\theta_{\sigma-t}\omega)|\, d\sigma \\
&\le \int_0^t \|\Xi(\sigma - t, 0, \omega) B(\theta_{\sigma-t}\omega)\| \cdot r(\theta_{\sigma-t}\omega)\, d\sigma \\
&\le \int_{-\infty}^\infty \|\Xi(\sigma, 0, \omega) B(\theta_\sigma\omega)\| \cdot r(\theta_\sigma\omega)\, d\sigma, \quad \forall t \ge 0, \quad \tilde{\forall}\omega \in \Omega.
\end{aligned}$$

The argument repeatedly applied above shows that the map $r_2\colon \Omega \to \mathbb{R}_{\ge 0}$ defined by

$$r_2(\omega) := \int_{-\infty}^\infty \|\Xi(\sigma, 0, \omega) B(\theta_\sigma\omega)\| \cdot r(\theta_\sigma\omega)\, d\sigma, \quad \omega \in \Omega,$$

is tempered. Now $r_1 + r_2$ is tempered and we have

$$|\check{\xi}_t^{x,u}(\omega)| = |\varphi(t, \theta_{-t}, x(\theta_{-t}\omega), u)| \le r_1(\omega) + r_2(\omega), \quad \forall t \ge 0, \quad \tilde{\forall}\omega \in \Omega.$$

This proves that $\xi^{x,u}$ is tempered. Since u tempered and x tempered were chosen arbitrarily, this completes the proof that φ is a tempered cocycle.

Remark 2.2. If $\|A(\cdot)\| \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, the largest eigenvalue $\overline{\lambda}(\cdot)$ of the Hermitian part of $A(\cdot)$ is such that

$$\mathbb{E}\overline{\lambda} := \int_\Omega \overline{\lambda}(\omega)\, d\mathbb{P}(\omega) < 0,$$

and the underlying MPDS θ is ergodic, then it follows from [5, p. 60, Theorem 2.1.2] that (L2) holds with $\lambda := -(\mathbb{E}\overline{\lambda} + \varepsilon)$ for any choice of $\varepsilon \in (0, -\mathbb{E}\overline{\lambda})$. □

2.4 Monotone RDSI's

Suppose that $(X, \leq)$ is a partially ordered space. For any $a, b \in X^\Omega_\mathscr{B}$, we write $a \leq b$ to mean that $a(\omega) \leq b(\omega)$ for θ-almost all $\omega \in \Omega$. Similarly, for any $p, q \in \mathscr{S}^X_\theta$, we write $p \leq q$ to mean that $p(t,\omega) \leq q(t,\omega)$ for all $t \geq 0$, for θ-almost all $\omega \in \Omega$. Observe that this convention naturally induces partial orders in $X^\Omega_\mathscr{B}$ and $\mathscr{S}^X_\theta$.

Definition 2.13 (Monotone RDSI). An RDSI $(\theta, \varphi, \mathscr{U})$ is said to be *monotone* if the underlying state and input spaces are partially ordered spaces $(X, \leq_X)$, $(U, \leq_U)$, and

$$\varphi(\cdot,\cdot,x(\cdot),u) \leq_X \varphi(\cdot,\cdot,z(\cdot),v)$$

whenever $x, z \in X^\Omega_\mathscr{B}$ and $u, v \in \mathscr{U}$ are such that $x \leq_X z$ and $u \leq_U v$.

In particular, if

$$\varphi(t,\omega,x,u) \leq_X \varphi(t,\omega,z,v)$$

holds for every $t \geq 0$, every $\omega \in \Omega$, and every $x, z \in X$ and $u, v \in \mathscr{U}$ such that $x \leq_X z$ and $u \leq_U v$, then it follows that $(\theta, \varphi, \mathscr{U})$ is monotone as per definition above.

Most of the time the underlying partially ordered space will be clear from the context. So unless there is any risk of confusion, we shall often drop the indices in "$\leq_X$" and "$\leq_U$," and write simply "$\leq$."

Proposition 2.5. *If an RDSI $(\theta, \varphi, \mathscr{U})$ is monotone and has an i/s characteristic $\mathscr{K}: U^\Omega_\theta \to X^\Omega_\theta$, then $\mathscr{K}$ is order-preserving; in other words, if $u, v \in U^\Omega_\theta$ and $u \leq v$, then $\mathscr{K}(u) \leq \mathscr{K}(v)$.*

Proof. The proof is straightforward, and we emphasize its main purpose of pointing out a subtlety in Definition 2.13 which might have otherwise gone overlooked. Pick any $u, v \in U^\Omega_\theta$ such that $u \leq v$, and fix $x \in X^\Omega_\theta$ arbitrarily. Then $x \leq x$, and $\bar{u} \leq \bar{v}$.

By Definition 2.13, there exists a θ-invariant subset of full-measure $\tilde{\Omega} \subseteq \Omega$ such that

$$\varphi(t,\omega,x(\omega),\bar{u}) \le \varphi(t,\omega,x(\omega),\bar{v}), \quad \forall t \ge 0,\ \forall \omega \in \tilde{\Omega}.$$

Thus

$$\varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega),\bar{u}) \le \varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega),\bar{v}), \quad \forall t \ge 0,\ \forall \omega \in \tilde{\Omega},$$

in view of the θ-invariance of $\tilde{\Omega}$. The result then follows by taking the limit as $t \to \infty$ on both sides of the inequality above for each fixed $\omega \in \tilde{\Omega}$. (Recall that, from the definition of i/s characteristic, such limits exist for θ-almost all $\omega \in \Omega$.) □

2.4.1 Converging Input to Converging State

The "converging input to converging state" result below was first stated and proved for deterministic and finite-dimensional "monotone control systems" by Angeli and Sontag [1, Proposition V.5(2)]. In [10, Theorem 1], Enciso and Sontag explore normality to extend the result to infinite-dimensional systems. Replacing the geometric properties in [10] by minihedrality and adding a compactness assumption it is possible to extend this result to monotone RDSI's.

Recall that a (closed) *cone* in a vector space X is a subset $X_+ \subseteq X$ such that $X_+ + X_+ \subseteq X$, $cX_+ \subseteq X_+$ for every $c \ge 0$, and $X_+ \cap (-X_+) = \varnothing$. The cone X_+ induces a partial order $\le_X$ in X, defined by

$$x \le_X y \quad \Leftrightarrow \quad y - x \in X_+.$$

A cone is said to be *solid* if it has nonempty interior, and *minihedral* if every finite subset has a supremum. If X is a normed space, then X_+ is said to be *normal* if there exists a constant $k \ge 0$ such that $0 \le x \le y$ implies $\|x\| \le k\|y\|$.

Theorem 2.3 (Random CICS). *Suppose that X and U are separable Banach spaces, partially ordered by solid, normal, minihedral cones $X_+ \subseteq X$ and $U_+ \subseteq U$, respectively. Let $(\theta,\varphi,\mathscr{U})$ be a tempered, monotone RDSI with state space X and input space U, and suppose that φ has a continuous i/s characteristic $\mathscr{K}: U_\theta^\Omega \to X_\theta^\Omega$. If $u \in \mathscr{U}$ and $u_\infty \in U_\theta^\Omega$ are such that*

(i) u is tempered and eventually precompact, and
(ii) $\check{u}_t \longrightarrow_\theta u_\infty$ as $t \to \infty$,

then

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathscr{K}(u_\infty) \quad as \quad t \to \infty, \quad \forall x \in X_\theta^\Omega. \tag{2.28}$$

In other words, if the pullback trajectories of u are eventually precompact and converge to u_∞ in the tempered sense, then the pullback trajectories of φ subject to u and starting at any tempered random state x will converge to $\mathscr{K}(u_\infty)$ in the tempered sense as well.

Proof. Fix arbitrarily $x \in X_\theta^\Omega$. From (i), u is tempered. Since φ is assumed to be tempered, the θ-stochastic process $\xi^{x,u}$ is also tempered (see Definition 2.11). In particular,

$$\|\check{\xi}_t^{x,u}(\cdot) - (\mathscr{K}(u_\infty))(\cdot)\| \leq \|\check{\xi}_t^{x,u}(\cdot)\| + \|(\mathscr{K}(u_\infty))(\cdot)\|,$$

which is in turn bounded by a nonnegative tempered random variable for large enough values of $t \geq 0$. Thus in order to prove the tempered convergence in (2.28), it remains to show the pointwise convergence; in other words, we need only show that

$$\check{\xi}_t^{x,u}(\omega) \longrightarrow \big(\mathscr{K}(u_\infty)\big)(\omega) \quad \text{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega. \tag{2.29}$$

This will require some setting up.

Since U_+ is solid and normal, it follows from temperedness and Proposition [5, p. 89, Proposition 3.2.2] that there exist a tempered random variable $v\colon \Omega \to \operatorname{int} U_+$ and a $t_u \geq 0$ such that

$$\check{u}_t(\omega) \in [-v(\omega), v(\omega)], \quad \forall t \geq t_u, \quad \tilde{\forall}\omega \in \Omega.$$

Moreover, $[-v, v]$ is a random closed set by Proposition [5, p. 88, Proposition 3.2.1](1); in particular, it is a random set. So $[-v, v]$ is indeed a tempered random set—temperedness follows from normality. In view of the assumption (i) that u is eventually precompact, by picking a larger t_u, if necessary, we may assume without loss of generality that $\beta_u^{t_u}(\omega)$ is precompact for θ-almost all $\omega \in \Omega$.

Let $(a_\tau)_{\tau \geq t_u}$ and $(b_\tau)_{\tau \geq t_u}$ be, respectively, *lower* and *upper tails* of the pullback trajectories of u:

$$a_\tau(\omega) := \inf_{t \geq \tau} u_t(\theta_{-t}\omega) = \inf \beta_u^\tau(\omega), \quad \tau \geq t_u,$$

and

$$b_\tau(\omega) := \sup_{t \geq \tau} u_t(\theta_{-t}\omega) = \sup \beta_u^\tau(\omega), \quad \tau \geq t_u,$$

for each $\omega \in \Omega$ such that $\beta_u^{t_u}(\omega)$ is precompact. It follows from the hypotheses that U is separable and U_+ is minihedral that the lower and upper tails of the pullback trajectories of u are well-defined, and the maps $\omega \mapsto a_\tau(\omega)$, $\omega \in \Omega$, and $\omega \mapsto b_\tau(\omega)$, $\omega \in \Omega$, are measurable for each $\tau \geq t_u$ (see [5, Theorem 3.2.1, p. 90]). For each $\tau \geq t_u$, we have $a_\tau, b_\tau \in [-v, v]$. Thus by normality a_τ, b_τ are indeed tempered random variables. Moreover,

$$a_\tau, b_\tau \longrightarrow_\theta u_\infty \quad \text{as} \quad \tau \to \infty, \tag{2.30}$$

which also follows by normality.

For each $\tau \geq t_u$, let $\bar{a}_\tau, \bar{b}_\tau$ be the θ-stationary processes generated by a_τ, b_τ, respectively. Then

$$(\bar{a}_\tau)_s(\omega) = a_\tau(\theta_s\omega) = \inf_{t\geq\tau} u_t(\theta_{-t}\theta_s\omega) \leq u_{\tau+s}(\theta_{-(\tau+s)}\theta_s\omega) = (\rho_\tau(u))_s(\omega)$$

and, similarly,

$$(\rho_\tau(u))_s(\omega) \leq (\bar{b}_\tau)_s(\omega), \quad \forall \tau \geq t_u, \quad \forall s \geq 0, \quad \forall \omega \in \Omega.$$

Thus

$$\bar{a}_\tau \leq \rho_\tau(u) \leq \bar{b}_\tau, \quad \forall \tau \geq t_u. \tag{2.31}$$

We now return to (2.29). Using the cocycle property, we may rewrite

$$\begin{aligned}\check{\xi}_t^{x,u}(\omega) &= \varphi(t-\tau, \theta_{-(t-\tau)}\omega, \varphi(\tau, \theta_{-t}\omega, x(\theta_{-t}\omega), u), \rho_\tau(u)) \\ &= \varphi(t-\tau, \theta_{-(t-\tau)}\omega, x_\tau(\theta_{-(t-\tau)}\omega), \rho_\tau(u)) \\ &= \check{\xi}_{t-\tau}^{x_\tau,\rho_\tau(u)}(\omega), \quad \forall \omega \in \Omega, \quad \forall t \geq \tau \geq t_u,\end{aligned}$$

where $x_\tau \in X_\theta^\Omega$ is defined by $x_\tau := \check{\xi}_\tau^{x,u}$. Therefore

$$\|\check{\xi}_{\tau+s}^{x,u}(\omega) - (\mathscr{K}(u_\infty))(\omega)\| = \|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - (\mathscr{K}(u_\infty))(\omega)\|$$

for every $\omega \in \Omega$, for all $s \geq 0$, for all $\tau \geq t_u$. For any such ω, s, τ, we have

$$\begin{aligned}\|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \leq\ & \|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - \check{\xi}_s^{x_\tau,\bar{a}_\tau}(\omega)\| \\ & + \|\check{\xi}_s^{x_\tau,\bar{a}_\tau}(\omega) - (\mathscr{K}(a_\tau))(\omega)\| \\ & + \|(\mathscr{K}(a_\tau))(\omega) - (\mathscr{K}(u_\infty))(\omega)\|.\end{aligned}$$

From (2.30) and the continuity of $\mathscr{K}$, there exist θ-invariant subsets $\tilde{\Omega}_a$ and $\tilde{\Omega}_b$ of full measure of Ω such that

$$\|(\mathscr{K}(a_\tau))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \longrightarrow 0, \quad \text{as} \quad \tau \to \infty, \quad \forall \omega \in \tilde{\Omega}_a,$$

and

$$\|(\mathscr{K}(b_\tau))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \longrightarrow 0, \quad \text{as} \quad \tau \to \infty, \quad \forall \omega \in \tilde{\Omega}_b.$$

Similarly, from the definition of i/s characteristic, for any integer $n \geq t_u$, there exist θ-invariant subsets $\tilde{\Omega}_{a,n}$ and $\tilde{\Omega}_{b,n}$ of full measure of Ω such that

$$\|\check{\xi}_s^{x_n,\bar{a}_n}(\omega) - (\mathscr{K}(a_n))(\omega)\| \longrightarrow 0, \quad \text{as} \quad s \to \infty, \quad \forall \omega \in \tilde{\Omega}_{a,n},$$

and

$$\|\check{\xi}_s^{x_n,\bar{b}_n}(\omega) - (\mathscr{K}(b_n))(\omega)\| \longrightarrow 0, \quad \text{as} \quad s \to \infty, \quad \forall \omega \in \tilde{\Omega}_{b,n}.$$

Now by (2.31) and monotonicity, for each integer $n \geq t_u$, there exists a θ-invariant subset of full measure $\tilde{\Omega}_{\leq,n} \subseteq \Omega$ such that

$$\check{\xi}_s^{x_n,\bar{a}_n}(\omega) \leq \check{\xi}_s^{x_n,\rho_n(u)}(\omega) \leq \check{\xi}_s^{x_n,\bar{b}_n}(\omega), \quad \forall s \geq 0, \quad \forall \omega \in \tilde{\Omega}_{\leq,n}.$$

Let[8]

$$\tilde{\Omega} := \tilde{\Omega}_a \cap \tilde{\Omega}_b \cap \left(\bigcap_{n=\lceil t_u \rceil}^{\infty} \tilde{\Omega}_{a,n} \right) \cap \left(\bigcap_{n=\lceil t_u \rceil}^{\infty} \tilde{\Omega}_{b,n} \right) \cap \left(\bigcap_{n=\lceil t_u \rceil}^{\infty} \tilde{\Omega}_{\leq,n} \right).$$

Thus $\tilde{\Omega}$ is a countable intersection of θ-invariant subsets of full measure of Ω and, hence, itself a θ-invariant subset of full measure of Ω. We shall show that convergence in (2.29) occurs for every $\omega \in \tilde{\Omega}$.

Fix arbitrarily an $\omega \in \tilde{\Omega}$ and a positive integer k. It follows from the construction of $\tilde{\Omega}$ that there exists an integer $n_k \geq t_u$ such that

$$\|(\mathscr{K}(a_\tau))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| < 1/k, \quad \forall \tau \geq n_k,$$

and

$$\|(\mathscr{K}(b_\tau))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| < 1/k, \quad \forall \tau \geq n_k.$$

Now we can use the convergence in the definition of i/s characteristic to choose an $s_k \geq 0$ such that

$$\|\check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega) - (\mathscr{K}(a_{n_k}))(\omega)\| < 1/k, \quad \forall s \geq s_k,$$

and

$$\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - (\mathscr{K}(b_{n_k}))(\omega)\| < 1/k, \quad \forall s \geq s_k.$$

[8] For any $x \in \mathbb{R}$, we write $\lceil x \rceil$ to denote the smallest integer larger than or equal to x.

Again from the construction of $\tilde{\Omega}$, we have

$$\check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega) \leq \check{\xi}_s^{x_{n_k},\rho_{n_k}(u)}(\omega) \leq \check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega), \quad \forall s \geq 0.$$

Thus

$$\|\check{\xi}_s^{x_{n_k},\rho_{n_k}(u)}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \leq C\,\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\|, \quad \forall s \geq 0,$$

where $C \geq 0$ is the normality constant for U_+. Now

$$\begin{aligned}
\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| &\leq \|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - (\mathscr{K}(b_{n_k}))(\omega)\| \\
&\quad + \|(\mathscr{K}(b_{n_k}))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \\
&\quad + \|(\mathscr{K}(u_\infty))(\omega) - (\mathscr{K}(a_{n_k}))(\omega)\| \\
&\quad + \|(\mathscr{K}(a_{n_k}))(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \\
&\leq 4/k, \quad \forall s \geq s_k.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|\check{\xi}_t^{x,u}(\omega) - (\mathscr{K}(u_\infty))(\omega)\| &= \|\check{\xi}_{t-n_k}^{x_{n_k},\rho_{n_k}(u)}(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \\
&\leq \|\check{\xi}_{t-n_k}^{x_{n_k},\rho_{n_k}(u)}(\omega) - \check{\xi}_{t-n_k}^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \\
&\quad + \|\check{\xi}_{t-n_k}^{x_{n_k},\bar{a}_{n_k}}(\omega) - (\mathscr{K}(a_{n_k}))(\omega)\| \\
&\quad + \|(\mathscr{K}(a_{n_k}))(\omega) - (\mathscr{K}(u_\infty))(\omega)\| \\
&< 4C/k + 1/k + 1/k \\
&= (4C+2)/k, \quad \forall t \geq n_k + s_k.
\end{aligned}$$

Since $\omega \in \tilde{\Omega}$ and the positive integer k were chosen arbitrarily, this completes the proof. □

2.4.2 Cascades

We now discuss a few applications of the "converging input to converging state" theorem just proved. Separate work in preparation deals with a small-gain theorem for random dynamical systems, a brief outline of which will be given at the end of the chapter.

Let (θ, ψ) be an autonomous RDS evolving on a space $Z = X_1 \times X_2$. We say that (θ, ψ) is *cascaded* if the flow ψ can be decomposed as

$$\psi(t,\omega,(x_1(\omega),x_2(\omega))) \equiv \begin{pmatrix} \varphi_1(t,\omega,x_1(\omega)) \\ \varphi_2(t,\omega,x_2(\omega),(\eta_1)^{x_1}) \end{pmatrix},$$

for some RDSO (θ,φ_1,h_1) with state space X_1 and output space Y_1, and some RDSI $(\theta,\varphi_2,\mathscr{U}_2)$ with state space X_2, input space $U_2 = Y_1$, and set of θ-inputs $\mathscr{U}_2$ containing all (forward) output trajectories of (θ,φ_1,h_1). In this case we write $\psi = \varphi_1 \ltimes \varphi_2$. Recall from item (1) in Theorem 2.2 that if the generator of a discrete RDS can be decomposed as in (2.14), then this RDS is a cascade. A similar decomposition can be done for systems generated by RDEI's whose generator satisfies the natural analogues of (2.14).

Example 2.4 (Bounded Outputs). Let $(\theta,\psi) := (\theta,\varphi_1 \ltimes \varphi_2)$ be a cascaded RDS as above. Suppose that (θ,φ_1,h_1) is an RDSO evolving on a normed space X_1, and such that (θ,φ_1) has a unique, globally attracting equilibrium $(\xi_1)_\infty \in X_{\mathscr{B}}^{\Omega}$:

$$(\check{\xi}_1)_t^{x_1}(\omega) \longrightarrow (\xi_1)_\infty(\omega), \quad \text{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega,\ \forall x_1 \in (X_1)_{\mathscr{B}}^{\Omega}.$$

Now suppose that $(\theta,\varphi_2,\mathscr{U}_2)$ is an RDSI satisfying the hypotheses of Theorem 2.3, and that the output function h_1 is *bounded*; in other words, there exists $M \geq 0$ such that

$$\|h_1(\omega,x_1)\| \leq M, \quad \forall x_1 \in X_1,\ \tilde{\forall}\omega \in \Omega.$$

We prove that (θ,ψ) has a unique equilibrium which is attracting for all tempered random initial states.

By continuity of h with respect to the state variable, we have

$$\begin{aligned}(\check{\eta}_1)_t^{x_1}(\omega) &= h_1(\omega,(\check{\xi}_1)_t^{x_1}(\omega)) \\ &\longrightarrow h_1(\omega,(\xi_1)_\infty(\omega)), \quad \text{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega,\ \forall x_1 \in (X_1)_{\mathscr{B}}^{\Omega}.\end{aligned}$$

Since h_1 is bounded, the convergence and the limit are automatically tempered. Thus

$$\check{\xi}_t^z(\omega) \longrightarrow \begin{pmatrix} (\xi_1)_\infty(\omega) \\ \mathscr{K}((u_2)_\infty)(\omega) \end{pmatrix} \quad \text{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega,\ \forall z \in Z_\theta^{\Omega},$$

by Theorem 2.3. In particular, the convergence in the second coordinate is tempered.

For conditions guaranteeing that an RDS (θ,φ) would have a unique, globally attracting equilibrium in the sense above, see [4, Theorem 3.2]. The assumption that the output is bounded is very reasonable in biological applications, since there is often a cut off or saturation in the reading of the strength of a signal.

Before we consider the next example, we develop a stronger notion of regularity for output functions than continuity with respect to the state variable. We seek a property which preserves tempered convergence, and which we could check it holds in specific examples.

Definition 2.14 (Tempered Lipschitz). An output function $h\colon \Omega \times X \to Y$ is said to be *tempered Lipschitz* (with respect to a given MPDS θ) if there exists a tempered random variable $L \in (\mathbb{R}_{\geq 0})^{\Omega}_{\theta}$ such that

$$\|h(\omega, x_1) - h(\omega, x_2)\| \leq L(\omega)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \ \tilde{\forall}\omega \in \Omega.$$

We refer to L as a *Lipschitz random variable for h*.

For example, suppose that $X \subseteq \mathbb{R}^n$, and that $h\colon \Omega \times X \to \mathbb{R}^k$ is an output function such that $h(\omega, \cdot)$ is differentiable for all ω in a θ-invariant set of full measure $\tilde{\Omega} \subseteq \Omega$. If the norm of the Jacobian with respect to x,

$$\omega \longmapsto \|D_x h(\omega, \cdot)\| := \sup_{x \in X} |D_x h(\omega, x)|, \quad \omega \in \Omega,$$

is finite and tempered, then h is tempered Lipschitz.

Lemma 2.5. *Let $h\colon \Omega \times X \to Y$ be a tempered Lipschitz output function, $p \in \mathscr{S}^X_{\theta}$ be a θ-stochastic process in X, and let $p_\infty \in X^{\Omega}_{\mathscr{B}}$. Let $q\colon \mathscr{T}_{\geq 0} \times \Omega \to Y$ be the θ-stochastic process in Y defined by*

$$q_t(\omega) := h(\omega, p_t(\omega)), \quad (t, \omega) \in \mathscr{T}_{\geq 0} \times \Omega,$$

and $q_\infty \in Y^{\Omega}_{\mathscr{B}(Y)}$ be the random variable in Y defined by

$$q_\infty(\omega) := h(\omega, p_\infty(\omega)), \quad \omega \in \Omega.$$

If $p_t \to_\theta p_\infty$, then $q_t \to_\theta q_\infty$.

Proof. It follows from continuity with respect to $x \in X$ that

$$q_t(\omega) = h(\omega, p_t(\omega)) \longrightarrow h(\omega, p_\infty(\omega)) = p_\infty(\omega) \quad \text{as} \quad t \to \infty, \quad \tilde{\forall}\omega \in \Omega.$$

Now because $p_t \to_\theta p_\infty$, there exist $r \in (\mathbb{R}_{\geq 0})^{\Omega}_{\theta}$ and $t_0 \geq 0$ such that

$$\|p_t(\omega) - p_\infty(\omega)\| \leq r(\omega), \quad \forall t \geq t_0, \ \tilde{\forall}\omega \in \Omega.$$

Let L be a Lipschitz random variable for h. Then

$$\begin{aligned}\|q_t(\omega) - q_\infty(\omega)\| &= \|h(\omega, p_t(\omega)) - h(\omega, p_\infty(\omega))\| \\ &\leq L(\omega)\|p_t(\omega) - p_\infty(\omega)\| \\ &\leq L(\omega)r(\omega), \qquad \forall t \geq t_0, \ \tilde{\forall}\omega \in \Omega.\end{aligned}$$

By item (3) in Lemma 2.4, Lr is tempered, completing the proof. □

Now suppose that $(\theta, \psi, \mathscr{U})$ is an RDSI evolving on a state space $Z = X_1 \times X_2$. In this case we say that $(\theta, \psi, \mathscr{U})$ is *cascaded* if the flow ψ can be decomposed as

$$\psi(t, \omega, (x_1(\omega), x_2(\omega)), u) \equiv \begin{pmatrix} \varphi_1(t, \omega, x_1(\omega), u) \\ \varphi_2(t, \omega, x_2(\omega), (\eta_1)^{x_1,u}) \end{pmatrix},$$

for some RDSIO $(\theta, \varphi_1, \mathscr{U}_1, h_1)$ with state space X_1, set of θ-inputs $\mathscr{U}_1 = \mathscr{U}$ and output space Y_1, and some RDSI $(\theta, \varphi_2, \mathscr{U}_2)$ with state space X_2, input space $U_2 = Y_1$, and set of θ-inputs $\mathscr{U}_2$ containing all (forward) output trajectories of $(\theta, \varphi_1, \mathscr{U}_1, h_1)$. In this case we also write $\psi = \varphi_1 \ltimes \varphi_2$. Item (1) in Theorem 2.2 can be generalized to contemplate this kind of cascades for discrete systems, as well as systems generated by random differential equations.

Example 2.5 (Tempered Lipschitz Outputs). Suppose that $(\theta, \varphi_1, \mathscr{U}_1)$ and $(\theta, \varphi_2, \mathscr{U}_2)$ in the decomposition above satisfy both the hypotheses of Theorem 2.3. If the output function h_1 is Lipschitz continuous, then $(\theta, \psi, \mathscr{U})$ also has the "converging input to converging state" property; that is, if $u \in \mathscr{U}$ is such that $\check{u}_t \to_\theta u_\infty$ for some $u_\infty \in U_\theta^\Omega$, then there exists a $\xi_\infty \in Z_\theta^\Omega$ such that

$$\check{\xi}_t^{z,u} \longrightarrow_\theta \xi_\infty, \quad \forall z \in Z_\theta^\Omega, \tag{2.32}$$

as well.

To see this, let $\mathscr{K}_1 \colon (U_1)_\theta^\Omega \to (X_1)_\theta^\Omega$ and $\mathscr{K}_2 \colon (U_2)_\theta^\Omega \to (X_2)_\theta^\Omega$ be the i/s characteristics of $(\theta, \varphi_1, \mathscr{U}_1)$ and $(\theta, \varphi_2, \mathscr{U}_2)$, respectively. Fix

$$z = (x_1, x_2) \in Z_\theta^\Omega = (X_1)_\theta^\Omega \times (X_2)_\theta^\Omega$$

arbitrarily. From Theorem 2.3, we have

$$(\check{\xi}_1)_t^{x_1,u} \longrightarrow_\theta \mathscr{K}_1(u_\infty).$$

Since h_1 is tempered Lipschitz, it follows from Lemma 2.5 that

$$(\check{\eta}_1)_t^{x_1,u} \longrightarrow_\theta (u_2)_\infty,$$

where

$$(u_2)_\infty := h_1(\cdot, \mathscr{K}_1(u_\infty)(\cdot)).$$

It follows, again from Theorem 2.3, that

$$(\check{\xi}_2)_t^{x_2,(\eta_1)^{x_1,u}} \longrightarrow_\theta \mathscr{K}_2((u_2)_\infty).$$

Hence

$$\check{\xi}_t^{z,u} = \begin{pmatrix} (\check{\xi}_1)_t^{x_1,u} \\ (\check{\xi}_2)_t^{x_2,(\eta_1)^{x_1,u}} \end{pmatrix} \longrightarrow_\theta \begin{pmatrix} \mathscr{K}_1(u_\infty) \\ \mathscr{K}_2((u_2)_\infty) \end{pmatrix}.$$

Since $z \in Z_\theta^\Omega$ was picked arbitrarily, this establishes (2.32).

The procedure above can be generalized to cascades of three or more systems to show that the "converging input to converging state" property will hold provided that it holds for its individual components—and the intermediate outputs are tempered Lipschitz. In Example 2.3, suppose we assume, in addition, that the off-diagonal entries of A and all entries of B are nonnegative θ-almost everywhere. Then the RDSI generated by the RDEI in the example is monotone and thus satisfies the hypotheses of Theorem 2.3. Tempered Lipschitz output functions are not difficult to come by, as we pointed out above. This yields a class of cascaded systems having the "converging input to converging state" property.

A couple more remarks about this example are in order. A cascade of monotone systems need not itself be monotone. So the construction above provides us with a way of checking the "converging input to converging state" property for systems which do not directly satisfy the hypotheses of Theorem 2.3. But even if it would be possible to check it directly that $(\theta, \varphi, \mathscr{U})$ already satisfies the hypotheses of Theorem 2.3, it might be easier to check them for each component—for instance, if $(\theta, \varphi, \mathscr{U})$ can be decomposed as a cascade of linear systems linked by (possibly nonlinear) tempered Lipschitz output functions.

We have illustrated in Examples 2.4 and 2.5 how one may obtain global convergence results for systems decomposable into cascades, as discussed in the Introduction. Further work in preparation deals with "closed loop" systems, and how "converging input to converging state" property can be used to prove small-gain theorems for such systems. Below we provide a brief outline of the idea.

2.4.3 Small-Gain Theorem

A small-gain theorem for the closed-loop of monotone RDSIO's with anti-monotone outputs follows along the lines of the deterministic case [1, 10]. Assuming the input and output spaces coincide, one defines an "input to output characteristic" $\mathscr{K}^Y : U_\theta^\Omega \to U_\theta^\Omega$ by composing the i/s characteristic (assuming, of course the underlying RDSI has one) with the output function h in the natural way:

$$\big(\mathscr{K}^Y(u)\big)(\omega) := h(\omega, \big(\mathscr{K}(u)\big)(\omega)), \quad u \in U_\theta^\Omega, \quad \omega \in \Omega.$$

If the iterates $(\mathscr{K}^Y)^{(k)}(u) := (\mathscr{K}^Y \circ \cdots \circ \mathscr{K}^Y)(u)$ (k times) of $\mathscr{K}^Y$ converge to a unique equilibrium u_∞ ("small-gain condition"), then every eventually precompact solution of the closed-loop system converges to $\mathscr{K}(u_\infty)$, the state characteristic corresponding to the input u_∞.

A proof as in [1, 10] goes by appealing to the random CICS property for monotone RDSI's above, after establishing a contraction property on the "limsup" and "liminf" (defined analogously as in these references) of external signals. Mild technical assumptions on the state and input/output spaces guarantee that said limsup's and liminf's are well-defined and measurable. Reasonable ("polynomial temperedness") growth conditions on the outputs guarantee that the input to output characteristic is well-defined as a map $U_\theta^\Omega \to U_\theta^\Omega$ (preserves temperedness). Separate work in preparation will provide all the details and several examples.

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