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A class of random control systems: Monotonicity and the convergent-input convergent-state property

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ABSTRACT

This paper introduces a new formalism for the study of random control systems. It develops the basic axiomatic framework and provides several basic definitions and results on equilibria and convergence. It also presents a *convergent-input convergent-state* (*CICS*) result, a key concept in the analysis of stability of feedback interconnections, for monotone systems.

I. INTRODUCTION

In the late 1980s, Ludwig Arnold conceived an elegant and deep approach to the foundations of random dynamics [1]. His paradigm of a *random dynamical system* (*RDS* for short) is based on an ultimately simple idea: view an RDS as consisting of two ingredients: a stochastic but autonomous "noise process" plus a classical dynamical system that is driven by this process. The noise process is described by a measure-preserving dynamical system. It is typically probabilistic, representing for example environmental perturbations, internal variability, randomly fluctuating parameters, model uncertainty, or measurement errors. The formalism allows nevertheless for deterministic periodic or almost-periodic driving processes as well. The resulting theory, developed since by many authors, provides a seamless integration of classical ergodic theory with modern dynamical systems, giving a theoretical framework parallel to classical smooth and topological dynamics (stability, attractors, bifurcation theory, and so forth) while allowing one to treat in a unified way the most important classes of dynamical systems with randomness, such as random differential or difference equations (basically, deterministic systems with randomly changing parameters) or stochastic ordinary and partial differential equations (white noise or more generally martingale-driven systems as studied in the Itô calculus).

The main goal of this paper is to propose a new RDS-based formalism for random control systems, that is, systems with inputs, which we call *random dynamical systems with inputs* (*RDSI*). Our motivation arises from the need to provide foundations for a constructive theory of interconnections and feedback for stochastic systems, one that will eventually generalize successful and widely applied deterministic approaches such as backstepping [2]. Of course, much excellent work has been done along these lines, not employing an RDS axiomatic approach, such as the studies [3], [4] on stochastic stabilization as well as [5], [6], [7] on feedback stabilization using noise to state stability analogs of *input to state stability* (*ISS*) [8], [2], [9], [10], Ideally, however, stochastic extensions of deterministic theory should take full advantage of the power of ergodic theory. For example, suppose that one wishes to study even something as simple as a scalar affine system $\dot{x} = ax + bu$, where a is not a constant but is randomly varying, $a = a(\omega)$. If $a(\omega) \leq -\lambda < 0$ for all ω, then stability will not be an issue. However, if all we have is that the expected value of a is negative, but $a(\omega)$ can take nonnegative values, then ergodic theory is needed in order to establish results on almost-sure stability (or convergence to equilibrium probability distributions). Thus we feel that an RDS-based theory is most natural in this context.

We first review classical RDS theory. This material is not new; however, with an eye to generalizations, we reformulate it in a slightly different language. We next define our new concept of RDSI, which extends the notion of RDS to systems in which there is an external input or forcing function which is itself a stochastic process. A major contribution of this work is the precise formulation of this concept, particularly the way in which the inputs are shifted in the semigroup (cocycle) property. Note that stochasticity of inputs is essential if one is to develop a theory of interconnected subsystems, as an input to one system in such an interconnection is typically obtained by combining the (necessarily random) outputs of other subsystems.

After establishing the basic framework, we turn to the question of convergent-input convergent-output (CICS) properties: when is it true that if an input converges to an equilibrium distribution, then solutions also do? Even for deterministic systems, CICS fails even for systems which are globally asymptotically stable with respect to constant inputs. This motivated, for deterministic systems, the introduction of the notions of ISS [8] and of monotone systems with inputs [11], either of which allows one to obtain CICS types of theorems. Fortunately, recent work by Chueshov [12] introduced the class of monotone RDS (without inputs), a theory that provides us with the tools needed to pursue the generalization of the latter to RDSI. Thus, we introduce also a class of monotone RDSI, and are able to formulate and prove a CICS theorem for monotone systems. A follow-up of this paper will introduce systems with (random) outputs and establish a small-gain theorem for monotone random dynamical systems with inputs and outputs, generalizing [11], which follows from the CICS tools developed here. Separate work in progress deals with generalizations of ISS within the same RDSI framework.

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II. Random Dynamical Systems

We first review the random dynamical systems framework of Arnold [1]. Suppose given a *measure-preserving dynamical system*[1] (*MPDS*) $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$; that is, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a topological group $(\mathcal{T}, +)$, and a measurable flow $(\theta_t)_{t\in\mathcal{T}}$ of measure-preserving maps $\Omega \to \Omega$ having properties (T1)–(T3):

- (T1) $(t,\omega) \mapsto \theta_t\omega$, $(t,\omega) \in \mathcal{T} \times \Omega$, is $(\mathcal{B}(\mathcal{T}) \otimes \mathcal{F})$-measurable[2];
- (T2) $\theta_{t+s} = \theta_t \circ \theta_s$ for every $t, s \in \mathcal{T}$ (semigroup property);
- (T3) $\mathbb{P} \circ \theta_t = \mathbb{P}$ for each $t \in \mathcal{T}$ (measure-preserving[3]).

A set $B \in \mathcal{F}$ is said to be θ-*invariant* if $\theta_t(B) = B$ for all $t \in \mathcal{T}$. The MPDS θ is said to be *ergodic* if, whenever $B \in \mathcal{F}$ is θ-invariant, then we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

In this paper, $\mathcal{T}$ will always refer to either $\mathbb{R}$ or $\mathbb{Z}$, depending on whether one is talking about continuous or discrete time, respectively. In either case, $\mathcal{T}_{\geqslant 0}$ denotes the nonnegative elements of $\mathcal{T}$.

In the context of MPDS's, it is often the case that a condition depending on $\omega \in \Omega$ is stated to be satisfied for all $\omega \in \widetilde{\Omega}$, for some θ-invariant $\widetilde{\Omega} \subseteq \Omega$ of full measure[4]. Most of the time it will not be necessary to specify said $\widetilde{\Omega}$ though. So we shall say 'for θ-almost all $\omega \in \Omega$' or write '$\widetilde{\forall}\omega \in \Omega$' to mean 'for all $\omega \in \widetilde{\Omega}$, for some θ-invariant set $\widetilde{\Omega} \subseteq \Omega$ of full measure'.

Let X be a metric space constituting the measurable space $(X, \mathcal{B})$ when equipped with the σ-algebra $\mathcal{B}$ of Borel subsets of X. A (*continuous*) *random dynamical system* (*RDS*) on X is a pair (θ, φ) in which θ is an MPDS and $\varphi : \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ is a (continuous) cocycle over θ; that is, a $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \Omega \otimes \mathcal{B})$-measurable map such that

- (S1) $\varphi(t,\omega) := \varphi(t,\omega,\cdot) : X \to X$ is continuous for every $t \geqslant 0$, $\omega \in \Omega$;
- (S2) $\varphi(0, w) = id_X$ for each $\omega \in \Omega$, and
$$\varphi(t+s,\omega) = \varphi(t, \theta_s\omega) \circ \varphi(s,\omega)$$
for every $s, t \geqslant 0$, and each $\omega \in \Omega$ (cocycle property).

Notice that RDS's include deterministic dynamical systems as the special case in which Ω is a singleton. The cocycle property generalizes the semigroup property of deterministic dynamical systems.

We now introduce a few pieces of terminology not found in Arnold [1] to facilitate the discussion.

[1]Some authors [12, Definition 1.1.1 on page 10],[1, page 635] refer to such an object primarily as a *metric dynamical system*. We find *measure-preserving*, which [1] also uses as a synonym, less confusing and more informative.

[2]In this paper, whenever S is a topological space, $\mathcal{B}(S)$ denotes the Borel σ-algebra of subsets S.

[3]Property (T3) is normally [13, Definition 1.1] stated as

$$\mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}(B), \quad \forall B \in \mathcal{F},\ \forall t \in \mathcal{T}.$$

But since it follows from (T2) that θ_t is invertible with $\theta_t^{-1} = \theta_{-t}$ for each $t \in \mathcal{T}$, the two formulations are equivalent in this context.

[4]In other words, $\theta_t\widetilde{\Omega} = \widetilde{\Omega}$ for all $t \in \mathcal{T}$, and $\mathbb{P}(\widetilde{\Omega}) = 1$.

In the context of RDS's, the analogue to points in the state space X for a deterministic system are random variables $\Omega \to X$, that is, $\mathcal{B}$-measurable maps $\Omega \to X$. We denote the set of all such maps by $X^\Omega_\mathcal{B}$. A θ-*stochastic process*[5] *on* X is a $(\mathcal{B} \otimes \mathcal{F})$-measurable map $q : \mathcal{T}_{\geqslant 0} \times \Omega \to X$. We denote $q_t := q(t,\cdot)$ for each $t \geqslant 0$. The set of all θ-stochastic process on X is denoted by $\mathcal{S}^X_\theta$.

Let (θ, φ) be an RDS. Given $x \in X^\Omega_\mathcal{B}$, we define the (*forward*) *trajectory starting at* x to be the θ-stochastic process $\xi^x : \mathcal{T}_{\geqslant 0} \times \Omega \to X$ defined by

$$\xi^x_t(\omega) := \varphi(t, \omega, x(\omega)), \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega. \tag{1}$$

The *pullback trajectory starting at* x is in turn defined to be the θ-stochastic process $\check{\xi}^x : \mathcal{T}_{\geqslant 0} \times \Omega \to X$ defined by

$$\check{\xi}^x_t(\omega) := \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)), \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega. \tag{2}$$

Note that $\check{\xi}^x_t(\omega) \equiv \xi^x_t(\theta_{-t}\omega)$.

We slightly modify the standard notion of equilibrium for RDS's (see, for instance, [12, Definition 1.7.1 on page 38]) to allow for the defining property to hold only θ-almost everywhere, as opposed to everywhere. So an *equilibrium* of an RDS is a random variable $x \in X^\Omega_\mathcal{B}$ such that

$$\xi^x_t(\omega) = x(\theta_t\omega), \quad \forall t \geqslant 0,\ \widetilde{\forall}\omega \in \Omega;$$

or, equivalently,

$$\check{\xi}^x_t(\omega) = x(\omega), \quad \forall t \geqslant 0,\ \widetilde{\forall}\omega \in \Omega.$$

In view of the notion of pullback convergence with which we will be working, the latter seems to be a more informative way to state the definition of equilibrium.

We discuss next an analogue, in the stochastic setting, of constant paths in the deterministic case. We start by defining a shift operator in $\mathcal{S}^X_\theta$. For each $s \geqslant 0$, let

$$\begin{array}{rcl} \rho_s : \mathcal{S}^X_\theta & \longrightarrow & \mathcal{S}^X_\theta \\ q & \longmapsto & \rho_s(q) \end{array} \tag{3}$$

be defined by

$$(\rho_s(q))_t(\omega) := q_{t+s}(\theta_{-s}\omega), \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega. \tag{4}$$

Definition 1 (θ-Stationary Processes): We say that a θ-stochastic process $\bar{q} \in \mathcal{S}^X_\theta$ is θ-*stationary* if $(\rho_s(\bar{q}))_t(\omega) = \bar{q}_t(\omega)$ for all $t, s \geqslant 0$, for θ-almost all $\omega \in \Omega$. △

Straightforward computations using the definition above show that a θ-stochastic process $\bar{q} \in \mathcal{S}^X_\theta$ is θ-stationary if, and only if there exists a random variable $q \in X^\Omega_\mathcal{B}$ such that

$$\bar{q}_t(\omega) = q(\theta_t\omega), \quad \forall t \geqslant 0,\ \widetilde{\forall}\omega \in \Omega. \tag{5}$$

In particular, if $\bar{q}$ is θ-stationary, then the corresponding random variable in (5) is given by $q := \bar{q}_0$. Thus it is uniquely determined θ-almost everywhere by $\bar{q}$. We shall always use an overbar to denote the θ-stationary θ-stochastic process $\bar{q}$ associated with a given random variable q, and vice versa.

[5]A 'θ-stochastic process' is indeed just a 'stochastic process' in the traditional sense. We use the prefix 'θ-' to emphasize the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and time semigroup $\mathcal{T}_{\geqslant 0}$ specified by the given MPDS.

III. RANDOM DYNAMICAL SYSTEMS WITH INPUTS

We now define a new concept. It extends the notion of RDS to systems in which there is an external input or forcing function. A contribution of this work is the precise formulation of this concept, particularly the way in which the argument of the input is shifted in the semigroup (cocycle) property.

As in the previous section, given a metric space U, we equip it with its Borel σ-algebra $\mathcal{B}(U)$ and denote by $U_{\mathcal{B}}^{\Omega}$ the set of Borel-measurable maps $\Omega \to U$. Let $\mathcal{S}_{\theta}^{U}$ be the set of all θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$. Given $u, v \in \mathcal{S}_{\theta}^{U}$ and $s \geqslant 0$, we define the θ-stochastic process $u \Diamond_s v \colon \mathcal{T}_{\geqslant 0} \times \Omega \to U$ by

$$(u \Diamond_s v)_\tau(\omega) = \begin{cases} u_\tau(\omega)\,, & 0 \leqslant \tau < s \\ v_{\tau - s}(\theta_s \omega)\,, & s \leqslant \tau \end{cases}\,,$$

for all $\tau \geqslant 0$, $\omega \in \Omega$.

Definition 2 (θ-Inputs): We say that a subset $\mathcal{U} \subseteq \mathcal{S}_{\theta}^{U}$ is a *set of θ-inputs* if it has the following three properties.

(1) Every $u \in \mathcal{S}_{\theta}^{U}$ such that $u(\cdot, \omega) : \mathcal{T}_{\geqslant 0} \to U$ is constant for every $\omega \in \Omega$ belongs to $\mathcal{U}$.

(2) $u \Diamond_s v \in \mathcal{U}$ for any $u, v \in \mathcal{U}$ and any $s \in \mathcal{T}_{\geqslant 0}$.

(3) If $u \in \mathcal{S}_{\theta}^{U}$ is such that, for every $s \in \mathcal{T}_{\geqslant 0}$, there exists a $v^{(s)} \in \mathcal{U}$ such that

$$u\big|_{[0,s)\times\Omega} = v^{(s)}\big|_{[0,s)\times\Omega}\,,$$

then $u \in \mathcal{U}$. △

In other words, a set of θ-inputs is a subset of $\mathcal{S}_{\theta}^{U}$ which 1) contains all the θ-stochastic processes which are constant along each $\omega \in \Omega$, 2) is closed under concatenation and 3) contains all θ-stochastic processes such that all their truncations are also in the set.

Definition 3 (RDSI): A *random dynamical system with inputs* (*RDSI*) is a triple $(\theta, \varphi, \mathcal{U})$ consisting of an MPDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathcal{T}})$, a set of θ-inputs $\mathcal{U} \subseteq \mathcal{S}_{\theta}^{U}$, and a map

$$\varphi : \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U} \to X$$

satisfying

(I1) $\varphi(\cdot, \cdot, \cdot, u) \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable for each fixed $u \in \mathcal{U}$;

(I1′) the map $\widetilde{\varphi} \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \times U \to X$ defined by

$$\widetilde{\varphi}(t, \omega, x, \tilde{u}) := \varphi(t, \omega, x, c(\tilde{u}))\,,$$

$(t, \omega, x, \tilde{u}) \in \mathcal{T}_{\geqslant 0} \times \Omega \times X \times U$, where $(c(\tilde{u}))_t(\omega) \equiv \tilde{u}$, is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}(U))$-measurable;

(I2) $\varphi(t, \omega, \cdot, u) : X \to X$ is continuous for each fixed $(t, \omega, u) \in \mathcal{T}_{\geqslant 0} \times \Omega \times \mathcal{U}$;

(I3) $\varphi(0, \omega, x, u) = x$ for each $(\omega, x, u) \in \Omega \times X \times \mathcal{U}$;

(I4) given $s, t \geqslant 0$, $\omega \in \Omega$, $x \in X$, $u, v \in \mathcal{U}$, if

$$\varphi(s, \omega, x, u) = y$$

and

$$\varphi(t, \theta_s \omega, y, v) = z\,,$$

then

$$\varphi(s + t, \omega, x, u \Diamond_s v) = z\,;$$

(I5) and given $t \geqslant 0$, $\omega \in \Omega$, $x \in X$, and $u, v \in \mathcal{U}$, if $u_\tau(\omega) = v_\tau(\omega)$ for almost all $\tau \in [0, t)$, then $\varphi(t, \omega, x, u) = \varphi(t, \omega, x, v)$. △

(I1)–(I2) are regularity conditions. (I3) means that nothing has "happened" if one is still at time $t = 0$. (I4) generalizes the cocycle property and (I5) states that the evolution of an RDS subject to an input u is, so to speak, independent of irrelevant random states. Example 1 below will illustrate this concept.

Remark 1: Notice that for each $s, t \geqslant 0$, $x \in X$, $\omega \in \Omega$,

$$\varphi(t + s, \omega, x, u) \;=\; \varphi(t, \theta_s \omega, \varphi(s, \omega, x, u), \rho_s(u))$$

for all $u \in \mathcal{U}$. This follows by (I4) with $v = \rho_s(u)$, which then renders $u \Diamond_s v = u$. □

Recalling the definitions stated in Equations (3)–(4),

$$(\rho_s(u))_t(\theta_s \omega) = u_{t+s}(\omega)\,;$$

the righthand side is the input as interpreted by an observer of the RDSI φ who started at time $t_1 = 0$, while the left-hand side is how someone who started observing the system at time $t_2 = s$ would describe it at time t $(+\, t_2)$. Following this interpretation, a θ-stationary input would then be an input which is observed to be just the same, regardless of when one started observing it.

IV. θ-STATIONARY INPUTS AND CHARACTERISTICS

The concept of RDSI subsumes that of an RDS, as we shall explain below. Denote the subset of $\mathcal{S}_{\theta}^{U}$ consisting of θ-stationary inputs by $\bar{\mathcal{S}}_{\theta}^{U}$. Let $(\theta, \varphi, \mathcal{U})$ be an RDSI, and suppose that $u \in \mathcal{U} \cap \bar{\mathcal{S}}_{\theta}^{U}$ is some θ-stationary input. We consider the map

$$\begin{array}{rccc} \varphi_u \colon & \mathcal{T}_{\geqslant 0} \times \Omega \times X & \longrightarrow & X \\ & (t, \omega, x) & \longmapsto & \varphi(t, \omega, x, u) \end{array}\,.$$

It follows from condition (I1) that φ_u is measurable, from (I2) that $\varphi_u(t, \omega, \cdot)$ is continuous for each $(t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega$, and from (I3) that $\varphi_u(0, \omega, \cdot) = id_X$ for every $\omega \in \Omega$. Using (I4) with Definition 1 and Remark 1 above, we can show that φ_u has the cocycle property θ-almost everywhere. Thus, if necessary, we may redefine φ_u on the θ-invariant set of measure zero where the cocycle property may not hold, obtaining[6] an RDS (θ, φ_u).

Definition 4 (Equilibria): Let $(\theta, \varphi, \mathcal{U})$ be an RDSI, and suppose that $\bar{\mu} \in \mathcal{U} \cap \bar{\mathcal{S}}_{\theta}^{U}$, with generating random variable μ. An *equilibrium associated to* $\bar{\mu}$ (or to μ) is any equilibrium x of the RDS (θ, φ_μ). The set of all equilibria associated to $\bar{\mu}$ is denoted as $\mathcal{E}(\bar{\mu})$ (or $\mathcal{E}(\mu)$).

For deterministic systems (when Ω is a singleton), when the set $\mathcal{E}(\bar{\mu})$ consists of a single globally attracting equilibrium, the mapping $\bar{\mu} \mapsto \mathcal{E}(\bar{\mu})$ is the object called the "input to state characteristic" in the literature on monotone i/o systems. We extend this notion to RDSI's. For reasons which will be illustrated in Example 1 below and become clearer in the proof of Theorem 1, further conditions on the convergence of the states need to be assumed.

[6] In the language of Arnold [1], φ_u is a *crude* cocycle which can be *perfected* into an *undistinguishable* cocycle, which we also denote φ_u.

In what follows, given an MPDS θ and a normed space $(X, \|\cdot\|)$, we denote by X_θ^Ω the space of tempered random variables $\Omega \to X$; that is, the space of $\mathcal{F}$-measurable maps $r\colon \Omega \to X$ such that

$$\sup_{s\in\mathcal{T}} \|r(\theta_s\omega)\| e^{-\gamma|s|} < \infty\,, \quad \forall\gamma > 0\,,\ \widetilde{\forall}\omega \in \Omega\,.$$

We observe that X_θ^Ω constitutes a module over the ring $\mathbb{R}_\theta^\Omega$ of real-valued, tempered random variables.

Definition 5 (Tempered Convergence): Let $(\xi_\alpha)_{\alpha\in A}$ be a net in $X_\mathcal{B}^\Omega$ and ξ_∞ any random variable in $X_\mathcal{B}^\Omega$. We say that $(\xi_\alpha)_{\alpha\in A}$ converges to ξ_∞ in the *tempered sense* if there exists a nonnegative, tempered random variable $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ and an $\alpha_0 \in A$ such that

(1) $\xi_\alpha(\omega) \to \xi_\infty(\omega)$ as $\alpha \to \infty$ for θ-almost all $\omega \in \Omega$, and
(2) $\|\xi_\alpha(\omega) - \xi_\infty(\omega)\| \leqslant r(\omega)$ for all $\alpha \geqslant \alpha_0$, for θ-almost all $\omega \in \Omega$.

In this case we denote $\xi_\alpha \to_\theta \xi_\infty$ (as $\alpha \to \infty$). △

Definition 6 (Tempered Continuity): A map

$$\mathcal{K}\colon \mathcal{U} \subseteq U_\mathcal{B}^\Omega \to X_\mathcal{B}^\Omega$$

is said do be *tempered continuous* if, whenever $(u_\alpha)_{\alpha\in A}$ is a net in $\mathcal{U}$ convergent to $u_\infty \in \mathcal{U}$ in the tempered sense, then $\mathcal{K}(u_\alpha) \to_\theta \mathcal{K}(u_\infty)$ as $\alpha \to \infty$ as well. △

Definition 7 (I/S Characteristic): An RDSI $(\theta, \varphi, \mathcal{U})$ is said to have an *input to state (i/s) characteristic* $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ if $U_\theta^\Omega \subseteq \mathcal{U}$ and

$$\breve{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u) \quad \text{as} \quad t \to \infty\,,$$

for every $x \in X_\theta^\Omega$, for every $u \in U_\theta^\Omega$. △

Example 1 (Affine RDEI's): Let $(\theta, \varphi, \mathcal{S}_\infty^U)$ be the RDSI generated by the affine *random differential equation with inputs* RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad u \in \mathcal{S}_\infty^U\,, \qquad (6)$$

where $X = \mathbb{R}^n$, $U = \mathbb{R}^k$, $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are random matrices such that

$$t \longmapsto A(\theta_t\omega)\,, \quad t \geqslant 0\,, \quad \text{and} \quad t \longmapsto B(\theta_t\omega)\,, \quad t \geqslant 0\,,$$

are locally essentially bounded for every $\omega \in \Omega$, and $\mathcal{S}_\infty^U \subseteq \mathcal{S}_\theta^U$ is the subset of θ-stochastic processes u such that

$$t \longmapsto u_t(\omega)\,, \quad t \geqslant 0\,,$$

are locally essentially bounded for every $\omega \in \Omega$ as well. Then indeed

$$\varphi(t, \omega, x, u) \equiv \Phi(t, \omega, x) + \Psi(t, \omega, u)\,,$$

where

$$\Phi(t, \omega, x) \equiv \Xi(0, t, \omega) \cdot x$$

and

$$\Psi(t, \omega, u) \equiv \int_0^t \Xi(\sigma, t, \omega) B(\theta_\sigma\omega) u_\sigma(\omega)\, d\sigma\,,$$

$\Xi(\cdot,\cdot,\omega)\colon \mathbb{R}\times\mathbb{R} \to M_{n\times n}(\mathbb{R})$ being the fundamental matrix solution of the homogeneous part of (6), namely, the linear differential equation

$$\dot{\xi} = A(\theta_t\omega)\xi\,, \quad t \geqslant 0\,,$$

for each $\omega \in \Omega$ (see Chueshov [12, Section 2.1, pages 59–60]).

Now suppose in addition that A, B are such that

(L1) B is tempered and
(L2) there exist a $\lambda > 0$ and a nonnegative, tempered random variable $\gamma \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$ such that

$$\|\Xi(s, s + r, \omega)\| \leqslant \gamma(\theta_s\omega)\, \mathrm{e}^{-\lambda r}$$

for all $s \in \mathbb{R}$, all $r \geqslant 0$, for θ-almost all $\omega \in \Omega$.

Then $(\theta, \varphi, \mathcal{S}_\infty^U)$ has a continuous input to state characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$. In fact, it can be shown that

$$\lim_{t\to\infty} \Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = 0$$

and

$$\lim_{t\to\infty} \Psi(t, \theta_{-t}\omega, \bar{u}) = \int_{-\infty}^0 \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma \qquad (7)$$

for every $x \in X_\theta^\Omega$, every $u \in U_\theta^\Omega$, and θ-almost all $\omega \in \Omega$. Moreover, the convergence in each of these limits is tempered and the righthand side of (7) defines a tempered Borel-measurable function of $\omega \in \Omega$. So

$$(\mathcal{K}(u))(\omega) \equiv \int_{-\infty}^0 \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma\,.$$

The map $\mathcal{K}$ so defined can also be shown to be continuous in the sense of Definition 6. The proofs follow from estimates using the temperedness hypotheses (L1) and (L2) together with the observation that

$$\omega \longmapsto \|b(\theta.\omega)\, \mathrm{e}^{-\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,, \quad \omega \in \Omega\,,$$

is a tempered random variable for any tempered random variable $b \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$, any $\gamma > 0$, and any $p \in [1, \infty]$. ◇

Remark 2: If $\|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, the largest eigenvalue $\overline{\lambda}(\cdot)$ of the Hermitian part of $A(\cdot)$ is such that

$$\mathbb{E}\overline{\lambda} := \int_\Omega \overline{\lambda}(\omega)\, d\mathbb{P}(\omega) < 0\,,$$

and the underlying MPDS θ is ergodic, then it follows from [12, Theorem 2.1.2, page 60] that (L2) holds with $\lambda := -(\mathbb{E}\overline{\lambda} + \epsilon)$ for any choice of $\epsilon \in (0, -\mathbb{E}\overline{\lambda})$. □

V. MONOTONE RDSI

Suppose that $(X, \leqslant)$ is a partially ordered metric space. For any $a, b \in X_\mathcal{B}^\Omega$, we write $a \leqslant b$ to mean that $a(\omega) \leqslant b(\omega)$ for θ-almost all $\omega \in \Omega$. Similarly, for any $p, q \in \mathcal{S}_\theta^X$, we write $p \leqslant q$ to mean that $p(t, \omega) \leqslant q(t, \omega)$ for all $t \in \mathcal{T}_{\geqslant 0}$, for θ-almost all $\omega \in \Omega$.

Definition 8: An RDSI $(\theta, \varphi, \mathcal{U})$ is said to be *monotone* if the underlying state and input spaces are partially ordered metric spaces $(X, \leqslant_X)$, $(U, \leqslant_U)$, respectively, and

$$\varphi(\cdot, \cdot, x, u) \leqslant_X \varphi(\cdot, \cdot, z, v)$$

whenever $x, z \in X$ and $u, v \in \mathcal{S}_\theta^U$ are such that $x \leqslant_X z$ and $u \leqslant_U v$. △

Most often the underlying partial order will be clear from the context and we shall use simply $\leqslant$ to denote either of $\leqslant_X$ or $\leqslant_U$.

In this work we are particularly interested in "cone-induced" partial orders. Recall that a *cone* in a real normed space V is a closed, convex subset $V_+ \subseteq V$ such that $aV_+ \subseteq V$ for all $a \geqslant 0$ and $V_+ \cap (-V_+) = \{0\}$. The underlying space V is partially ordered by the relation

$$x \leqslant_{V_+} y \quad \Longleftrightarrow \quad y - x \in V_+ .$$

Since V_+ is closed, this partial order is compatible with the topology of V; in other words, the inequality is preserved by limits. A cone V_+ said to be *solid* if its interior $\operatorname{int} V_+$ is nonempty. We say that V_+ is a *normal* cone if there exists a constant $C \geqslant 0$ (the *normality constant*) such that $0 \leqslant_{V_+} x \leqslant_{V_+} y$ implies $\|x\| \leqslant C\|y\|$. The cone V_+ is said to be *minihedral* if the induced partial order in V behaves in a way such that every finite subset $B \subseteq V$ has an infimum $\inf B$ and a supremum $\sup B$; that is, $\inf B \leqslant x$ for all $x \in B$ and, if $y \leqslant x$ for all $x \in B$, then $y \leqslant x$ (analogously for the supremum). Given $a, b \in V$, we define the *conic interval* $[a, b] \subseteq V$ to be the set of points $x \in V$ such that $a \leqslant x \leqslant b$.

Definition 9 (Eventual Temperedness): We say that a θ-stochastic process $u \in \mathcal{S}_\theta^V$ is *eventually tempered* if there exist a tempered random variable $\beta \in (V_+)_\theta^\Omega$ and a $t_0 \geqslant 0$ such that

$$\check{u}_t(\omega) \in [-\beta(\omega), \beta(\omega)] , \quad \forall t \geqslant t_0 ,$$

for θ-almost $\omega \in \Omega$. △

Definition 10 (Tempered RDSI): Let $(\theta, \varphi, \mathcal{U})$ be an RDSI evolving in a normed space X with inputs defined in a normed space U, both partially ordered by cones X_+ and U_+, respectively. We say that φ is *tempered* if the trajectories $\xi^{x,u}$ are eventually tempered for every tempered initial state $x \in X_\theta^\Omega$ and every eventually tempered input $u \in \mathcal{U}$. △

The RDSI in Example 1 is tempered, and also monotone with respect to the partial orders in $\mathbb{R}^n$ and $\mathbb{R}^k$ induced by their respective positive orthant cones.

Remark 3: (1) Suppose that $V_+ \subseteq V$ is a solid, normal cone. If the pullback trajectory of a θ-stochastic process $u \in \mathcal{S}_\theta^V$ is tempered-convergent (see Definition 5), then u is eventually tempered.

(2) Conversely, if V_+ is normal, $u \in \mathcal{S}_\theta^V$ is eventually tempered and the pullback trajectories $\check{u}.(\omega)$ converge for θ-almost all $\omega \in \Omega$, then said convergence is tempered. □

VI. Convergent-Input Convergent-State

Theorem 1 (CICS): Suppose that X, U are separable Banach spaces, partially ordered by solid, normal, strongly minihedral cones $X_+ \subseteq X$ and $U_+ \subseteq U$, respectively. Let $(\theta, \varphi, \mathcal{U})$ be a tempered, monotone RDSI with state space X and input space U and suppose that φ has a continuous i/s characteristic $\mathcal{K} \colon U_\theta^\Omega \to X_\theta^\Omega$. If $u \in \mathcal{U}$ and $u_\infty \in U_\theta^\Omega$ are such that

(i) the closure $\overline{\beta_u^t}$ of the tail of the pullback of u is a compact random set[7] for each $t \geqslant 0$ and

(ii) $\check{u}_t \longrightarrow_\theta u_\infty$ as $t \to \infty$,

then

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u_\infty) \quad \text{as} \quad t \to \infty , \quad \forall x \in X_\theta^\Omega . \tag{8}$$

In other words, if the pullback trajectories of u are precompact and converge to u_∞ in the tempered sense, then the pullback trajectories of φ subject to u and starting at any tempered random state x will converge to $\mathcal{K}(u_\infty)$ in the tempered sense as well.

Proof: From (ii) and Remark 3, we know that u is eventually tempered. Now fix arbitrarily $x \in X_\theta^\Omega$. Since φ is tempered, the θ-stochastic process $\xi^{x,u}$ is also eventually tempered. So it follows again from Remark 3 that, in order to prove the tempered convergence in (8), we need only show the pointwise convergence; in other words, we need only show that

$$\check{\xi}_t^{x,u}(\omega) \longrightarrow \big(\mathcal{K}(u_\infty)\big)(\omega) \quad \text{as} \quad t \to \infty , \quad \widetilde{\forall} \omega \in \Omega . \tag{9}$$

It follows from eventual temperedness that there exist a tempered random variable $\beta \colon \Omega \to U_+$ and a $t_0 \geqslant 0$ such that

$$\check{u}_t(\omega) \in [-\beta(\omega), \beta(\omega)] , \quad \forall t \geqslant t_0 , \quad \widetilde{\forall} \omega \in \Omega .$$

By normality, every measurable selection of $[-\beta, \beta]$ is tempered.

For each $\tau \geqslant 0$, define $a_\tau, b_\tau \colon \Omega \to U$ by

$$a_\tau(\omega) := \inf \overline{\beta_u^\tau(\omega)} = \inf_{t \geqslant \tau} u_t(\theta_{-t}\omega)$$

and

$$b_\tau(\omega) := \sup \overline{\beta_u^\tau(\omega)} = \sup_{t \geqslant \tau} u_t(\theta_{-t}\omega)$$

for each $\omega \in \Omega$. It follows from the compactness assumption in (i) and [12, Theorem 3.2.1, page 90] that a_τ, b_τ are well-defined and measurable. For $\tau \geqslant t_0$, we have $a_\tau, b_\tau \in [-\beta, \beta]$, so a_τ, b_τ are indeed tempered random variables. Moreover, it can be shown using (ii) and normality of U_+ that

$$a_\tau, b_\tau \longrightarrow_\theta u_\infty \quad \text{as} \quad \tau \to \infty . \tag{10}$$

For each $\tau \geqslant 0$, let $\bar{a}_\tau, \bar{b}_\tau$ be the θ-stationary processes generated by a_τ, b_τ, respectively. Then

$$\begin{aligned} (\bar{a}_\tau)_s(\omega) &= a_\tau(\theta_s\omega) \\ &= \inf_{t \geqslant \tau} u_t(\theta_{-t}\theta_s\omega) \\ &\leqslant u_{\tau+s}(\theta_{-(\tau+s)}\theta_s\omega) \\ &= (\rho_\tau(u))_s(\omega) \end{aligned}$$

and, similarly,

$$(\rho_\tau(u))_s(\omega) \leqslant (\bar{b}_\tau)_s(\omega) , \quad s, \tau \geqslant 0 , \quad \omega \in \Omega .$$

[7]In a separable Banach space U, a multifunction $D \colon \Omega \to 2^U \backslash \{\varnothing\}$ is a compact random set if, and only if $D(\omega)$ is compact for each $\omega \in \Omega$ and $\{\omega \in \Omega \, ; \; D(\omega) \cap F \neq \varnothing\}$ is $\mathcal{F}$-measurable for every closed subset $F \subseteq U$. (See [12, Proposition 1.3.1(iii), page 20].)

Thus

$$\bar{a}_\tau \leqslant \rho_\tau(u) \leqslant \bar{b}_\tau\,, \quad \tau \geqslant 0\,. \tag{11}$$

For any $\omega \in \Omega$ and any $t \geqslant \tau \geqslant t_0$, we have

$$\begin{aligned} \|\check{\xi}_t^{x,u}(\omega) - (\mathcal{K}(u_\infty))(\omega)\| &\leqslant \|\check{\xi}_t^{x,u}(\omega) - \check{\xi}_t^{x,\bar{a}_\tau}(\omega)\| \\ &+\|\check{\xi}_t^{x,\bar{a}_\tau}(\omega) - (\mathcal{K}(a_\tau))(\omega)\| \\ &+\|(\mathcal{K}(a_\tau))(\omega) - (\mathcal{K}(u_\infty))(\omega)\|\,. \end{aligned}$$

Given any $\epsilon > 0$, it follows from (11) plus the continuity of $\mathcal{K}$ that there exists $\tau_0 \geqslant t_0$ such that

$$\|(\mathcal{K}(a_\tau))(\omega) - (\mathcal{K}(u_\infty))(\omega)\|,$$
$$\|(\mathcal{K}(b_\tau))(\omega) - (\mathcal{K}(u_\infty))(\omega)\| < \epsilon\,, \quad \tau \geqslant \tau_0\,.$$

Now we can use the convergence in the definition of input to state characteristic to choose a $t_1 \geqslant 0$ such that

$$\|\check{\xi}_t^{x,\bar{a}_{\tau_0}}(\omega) - (\mathcal{K}(a_{\tau_0}))(\omega)\| < \epsilon\,, \quad t \geqslant t_1\,.$$

Using the cocycle property, we may abbreviate $s := t - \tau_0$ and compute

$$\begin{aligned} &\check{\xi}_t^{x,u}(\omega) \\ =\ &\varphi(s, \theta_{-s}\omega, \varphi(\tau_0, \theta_{-t}\omega, x(\theta_{-t}\omega), u), \rho_{\tau_0}(u)) \\ =\ &\varphi(s, \theta_{-s}\omega, x_1(\theta_{-s}\omega), \rho_{\tau_0}(u)) \\ =\ &\check{\xi}_s^{x_1,\rho_{\tau_0}(u)}(\omega)\,, \end{aligned}$$

where $x_1 \in X_\theta^\Omega$ is defined by

$$x_1(\omega) := \varphi(\tau_0, \theta_{-\tau_0}\omega, x(\theta_{-\tau_0}\omega), u)\,, \quad \omega \in \Omega\,.$$

Now by (11) and monotonicity,

$$\check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega) \leqslant \check{\xi}_s^{x_1,\rho_{\tau_0}(u)}(\omega) \leqslant \check{\xi}_s^{x_1,\bar{b}_{\tau_0}}(\omega)\,, \quad s \geqslant 0\,.$$

Let $C \geqslant 0$ be the normality constant for U_+. Then

$$\begin{aligned} &\|\check{\xi}_s^{x_1,\rho_{\tau_0}(u)}(\omega) - \check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega)\| \\ \leqslant\ &C\|\check{\xi}_s^{x_1,\bar{b}_{\tau_0}}(\omega) - \check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega)\| \\ \leqslant\ &C\|\check{\xi}_s^{x_1,\bar{b}_{\tau_0}}(\omega) - (\mathcal{K}(b_{\tau_0}))(\omega)\| \\ &+C\|(\mathcal{K}(b_{\tau_0}))(\omega) - (\mathcal{K}(u_\infty))(\omega)\| \\ &+C\|(\mathcal{K}(u_\infty))(\omega) - (\mathcal{K}(a_{\tau_0}))(\omega)\| \\ &+C\|(\mathcal{K}(a_{\tau_0}))(\omega) - \check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega)\| \\ \leqslant\ &C\|\check{\xi}_s^{x_1,\bar{b}_{\tau_0}}(\omega) - (\mathcal{K}(b_{\tau_0}))(\omega)\| \\ &+C\|(\mathcal{K}(a_{\tau_0}))(\omega) - \check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega)\| + 2C\epsilon\,, \end{aligned}$$

for every $s \geqslant 0$. Again from the definition of input to state characteristic, one can choose $s_0 \geqslant 0$ large enough so that

$$\|\check{\xi}_s^{x_1,\bar{b}_{\tau_0}}(\omega) - (\mathcal{K}(b_{\tau_0}))(\omega)\| < \epsilon$$

and

$$\|(\mathcal{K}(a_{\tau_0}))(\omega) - \check{\xi}_s^{x_1,\bar{a}_{\tau_0}}(\omega)\| < \epsilon$$

for all $s \geqslant s_0$. It then follows that

$$\|\check{\xi}_t^{x_0,u}(\omega) - (\mathcal{K}(u_\infty))(\omega)\| < (4C+2)\epsilon\,,$$

for $t \geqslant \max\{t_1, \tau_0 + s_0\}$. Since $\epsilon > 0$ was arbitrary, (9) holds. ■

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